MATH 222 : ANALYSIS II – MEASURE & INTEGRATION
SPRING 2023
HOMEWORK 2

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1. Let $(X, \mathcal{F}, \mu)$ be a measure space. Consider the collection of sets

$$\mathscr{N} := \{Z \subset X : \exists A \in \mathcal{F} \text{ such that } A \supset Z \text{ and } \mu(A) = 0\}.$$

We define the **completion** of $(X, \mathcal{F}, \mu)$ — denoted by $(X, \overline{\mathcal{F}}, \overline{\mu})$ — as follows:

- Define the collection of sets $\overline{\mathcal{F}}$ as:
$$\overline{\mathcal{F}} := \{B \cup Z : B \in \mathcal{F},\ Z \in \mathscr{N}\}.$$
- Define $\overline{\mu}(B \cup Z) := \mu(B)$, where B and Z are as in the above definition.

Show that $\overline{\mathcal{F}}$ is the σ-algebra generated by $\mathcal{F} \cup \mathscr{N}$. Secondly, show that the function $\overline{\mu}$ on $\overline{\mathcal{F}}$ is a measure.

Remark. $(X, \overline{\mathcal{F}}, \overline{\mu})$ is called the completion because it is, clearly, a complete measure space.

2. Show — by using aspects of the argument that shows that $m*$ is not countably additive — that there exists a subset of $\mathbb{R}$ that is not Lebesgue measurable (i.e., is not in $\mathscr{M}_1$).

3. Show that, given any Lebesgue-measurable set $E \subsetneq \mathbb{R}$ with $m(E) > 0$, there exists a set $A \subset E$ that is not Lebesgue measurable.

4. Fix $n \in \mathbb{Z}_+$. Let $E \subset \mathbb{R}^n$. Using the fact that

($*$) $E \in \mathscr{M}_n \Rightarrow$ given any $\varepsilon > 0$, there exists an open set $\Omega_\varepsilon \subset \mathbb{R}^n$ such that $\Omega_\varepsilon \supset E$ and $m^*(\Omega_\varepsilon \setminus E) < \varepsilon$

for any E such that $m^*(E) < \infty$, show that ($*$) also holds true when $m^*(E) = \infty$.

5. Fix $n \in \mathbb{Z}_+$. Show that any non-empty at most countable subset of $\mathbb{R}^n$ is Lebesgue measurable and that its Lebesgue measure is zero.

6. Prove the following:

Theorem (Carathéodory)**.** *Let X be an infinite set and let μ^* be an outer measure on X. Define*

$$M(\mu^*) := \{E \subset X : \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E) \text{ for every } A \in \mathscr{P}(X)\},$$

and set $\mu := \mu^*|_{M(\mu^*)}$. Then $(X, M(\mu^*), \mu)$ is a measure space.

7. (Large Cantor sets) Consider a sequence $\{\alpha_1, \alpha_2, \alpha_3, \dots\} \subset (0, 1)$. Construct a Cantor-like set as follows: Let $K_0 := [0, 1]$. For each $n = 1, 2, 3, \dots$, define:

$$K_n := \text{the set obtained by removing open intervals that form the middle } \alpha_n^{\text{th}} \text{ fraction of each connected component of } K_{n-1}.$$

Define

$$K := \bigcap_{n=1}^{\infty} K_n.$$

(a) Show that K has empty interior. (**Note:** You should know by now how to argue that K is non-empty!)

(b) Show that K is Lebesgue measurable.

(c) Show that $m(K) > 0$ if and only if $\sum_{j=1}^{\infty} \alpha_n < +\infty$.

Hint. You will need a standard observation about infinite products. The tricky part in solving (c) is an argument in classical analysis linking the latter observation with the series $\sum_{j=1}^{\infty} \alpha_n$.

Remark. We discussed in class the following result: *Fix $n \in \mathbb{Z}_+$. A set $E \subseteq \mathbb{R}^n$ is Lebesgue measurable if and only if, given any $\varepsilon > 0$, there exists an closed set $C_\varepsilon \subset \mathbb{R}^n$ such that $C_\varepsilon \subset E$ and $m^*(E \setminus C_\varepsilon) < \varepsilon$.* The above construction shows why the word "closed" cannot be replaced by the word "open" in the latter result.