MATH 222 : ANALYSIS II – MEASURE & INTEGRATION
SPRING 2023
HOMEWORK 3

Instructor: GAUTAM BHARALI **Assigned: JANUARY 31, 2023**

---

1. Complete the following outline to show that there exists a function that is continuous on $[0,1]$ but **not** $(\mathscr{M}_1, \mathscr{M}_1)$-measurable (or, more accurately, not $(\mathscr{M}_1|_{[0,1]}, \mathscr{M}_1)$-measurable, since the domain of the desired function is $[0,1]$). Let K denote the standard Cantor set, let $K_0 \supsetneq K_1 \supsetneq K_2 \supsetneq \cdots$ be the sequence of compact sets, with each K_n comprised of 2^n closed intervals, such that

$$K_n := \bigsqcup_{j=1}^{2^n} [\alpha_{n,j}, \beta_{n,j}], \quad \text{and} \quad K = \bigcup_{n=0}^{\infty} K_n.$$

Using the above notation, let $f_n : [0,1] \to [0,1]$, $n = 0, 1, 2, \dots$, be the piecewise-affine continuous functions defined in class in this connection.

- Show that there exists a continuous function $f : [0,1] \to [0,1]$ such that $f_n \longrightarrow f$ uniformly as $n \to \infty$.
- Writing $\Psi(x) = x + f(x)$, $x \in [0,1]$, explain why $g := \Psi^{-1}$ is continuous (you may assume **without** proof the easy fact that Ψ is strictly increasing).
- Show that $S := \Psi(K) \in \mathscr{M}_1$ and that $m(S) > 0$.
- Finally, show that g is not $(\mathscr{M}_1|_{[0,1]}, \mathscr{M}_1)$-measurable. (You will need to appeal to a problem in Homework 2.)

2. Make use of the discussion in the previous problem to deduce that the Borel σ-algebra $\mathscr{B}(\mathbb{R}) \subsetneq \mathscr{M}_1$.

3. Let $(X, \mathcal{F})$ be a measurable space and let $f : X \to [-\infty, +\infty]$. Show that the following are equivalent:

(i) f is measurable.

(ii) $\{x \in X : f(x) > a\} \in \mathcal{F}$ for each $a \in \mathbb{R}$.

4. Let $(X, \mathcal{F})$ be a measurable space and let $f, g : X \to [-\infty, +\infty]$ be two measurable functions. Fix some number $a \in \mathbb{R}$. Define

$$h(x) := \begin{cases} a, & \text{if } f(x) = -g(x) = +\infty \text{ OR } f(x) = -g(x) = -\infty, \\ f(x) + g(x), & \text{otherwise.} \end{cases}$$

Show that h is measurable.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone function. Show that f is $(\mathscr{B}(\mathbb{R}), \mathscr{B}(\mathbb{R}))$-measurable.

6. Let $(X, \mathcal{F})$ be a measurable space and let $\{f_n\}_{n \in \mathbb{Z}_+}$ be a sequence of measurable $\mathbb{R}$-valued functions. Show that the set of points in $\mathbb{R}$ at which $\{f_n\}_{n \in \mathbb{Z}_+}$ converges to a finite limit is measurable.

7. Let $(X, \mathcal{F})$ be a measurable space. For $A \in \mathscr{P}(X)$, let χ_A denote the characteristic function of A. Let

$$s(x) \ := \ \sum_{j=1}^{N} \alpha_j \chi_{E_j},$$

where $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$ and $E_1, \ldots, E_N \subseteq X$. Show that if $E_1, \ldots, E_N \in \mathcal{F}$, then s is measurable. Now, assume that:

- $\alpha_j \neq \alpha_k$ when $j \neq k$, and
- $E_1, \ldots, E_N$ are pairwise disjoint.

Prove that if f is measurable, then $E_1, \ldots, E_N \in \mathcal{F}$.