MATH 222 : ANALYSIS II – MEASURE & INTEGRATION
SPRING 2023
HOMEWORK 4

Instructor: GAUTAM BHARALI **Assigned: FEBRUARY 7, 2023**

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1. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $A \in \mathcal{F}$. Let $\phi : X \to \mathbb{R}$ be a simple non-negative measurable function. Show that

$$\int_A \phi \, d\mu = \int_X \phi \chi_A \, d\mu.$$

2. In this problem, we shall use the notation ${}^{(\text{elem.})}\!\int_X$ to denote the integral of a simple non-negative measurable function (SNNMF) to emphasise that you must use **only** the definition of the integral of a SNNMF in your proof. With this clarification, prove that if $(X, \mathcal{F}, \mu)$ is a measure space, ϕ and ψ are two SNNMFs on X, and $\psi \geq \phi$, then

$${}^{(\text{elem.})}\!\int_X \psi \, d\mu \geq {}^{(\text{elem.})}\!\int_X \phi \, d\mu.$$

3. Fix $n \in \mathbb{Z}_+$ and equip $\mathbb{R}^n$ with the Borel σ-algebra. Fix $a \in \mathbb{R}^n$ and consider the Dirac mass at a; denote it by δ_a. (This measure was introduced in Homework 1.) Let $f : \mathbb{R}^n \to [-\infty, +\infty]$ be a Borel-measurable function. Follow the 3-step construction of the Lebesgue integral to:

i) describe all $[-\infty, +\infty]$-valued Lebesgue-integrable functions (for the given measure space: i.e., $(\mathbb{R}^n, (\mathbb{R}^n), \delta_a)$); and

ii) derive a formula for $\int_{\mathbb{R}^n} f \, d\delta_a$ for any Lebesgue-integrable f.

4. Show that the Monotone Convergence Theorem can be deduced from Fatou's Lemma.

5. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f : X \to [0, +\infty]$ be measurable. Show that

$$\int_X f \, d\mu = 0 \iff f = 0 \text{ a.e.}$$

7. Let $(X, \mathcal{F}, \mu)$ be a measure space, let $A \in \mathcal{F}$, and let $f : X \to [-\infty, +\infty]$ be measurable. Show that f is Lebesgue-integrable on A if and only if

$$\int_A |f| \, d\mu < +\infty.$$

8. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $A, B \in \mathcal{F}$ such that $X = A \cup B$. Consider a function $f : X \to [-\infty, +\infty]$. Show that if $f|_A$ is measurable (i.e., with respect to the σ-algebra $(A, \mathcal{F}|_A, \mu|_A)$) and $f|_B$ is measurable, then f is a measurable function.

9. Prove that the conclusion of Fatou's Lemma holds true if the sequence of functions $\{f_n\}$ is as featured in its statement and, instead of $f := \liminf_{n\to\infty} f_n$, we are given that $f_n \longrightarrow f$ a.e.