MATH 222 : ANALYSIS II – MEASURE & INTEGRATION
SPRING 2023
HOMEWORK 5

Instructor: GAUTAM BHARALI **Assigned: FEBRUARY 14, 2023**

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1. Prove that the conclusion of Fatou's Lemma holds true if, instead of the sequence of functions $\{f_n\}$ featured in its statement, we are given that each f_n is a $[-\infty, \infty]$-valued measurable function such that $f_n \geq 0$ a.e.

Remark. We used the above statement in the proof of the Dominated Convergence Theorem and discussed how one may prove it. Now, provide the details.

2. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f : X \to [-\infty, +\infty]$ be integrable. Show that f is integrable on A for every $A \in \mathcal{F}$.

3. Let $(X, \mathcal{F}, \mu)$ be a measure space and suppose $A \in \mathcal{F}$. Let $f : X \to [-\infty, +\infty]$ be a measurable function and suppose f is Lebesgue integrable on A. Show that the Lebesgue integral $\int_A f\, d\mu$ as defined in class agrees with the the integral of $f|_A$ that is obtained by:

- starting with the measure space $(A, \mathcal{F}|_A, \mu|_A)$ and viewing $f|_A$ as a measurable function relative to $(A, \mathcal{F}|_A)$; and
- retracing the 3-step construction of the Lebesgue integral by applying it to measure space $(A, \mathcal{F}|_A, \mu|_A)$.

4. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f : X \to [-\infty, +\infty]$ be integrable. Show that for any $c \in \mathbb{R}$, cf is integrable and that

$$\int_A (cf)\, d\mu = c \int_A f\, d\mu.$$

5. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f : X \to [-\infty, +\infty]$ be integrable. Show that $\mu(f^{-1}\{+\infty\}) = \mu(f^{-1}\{-\infty\}) = 0$.

6. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of non-negative measurable functions on X. Suppose there exists a function $f : X \to [0, +\infty]$ such that $f_n \longrightarrow f$ a.e. and $f \geq f_n$ a.e. for each $n \in \mathbb{Z}_+$. Show that

$$\lim_{n\to\infty} \int_X f_n\, d\mu = \int_X f\, d\mu.$$

Note. It is **not** given that f is integrable, due to which one cannot simply invoke the DCT.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function with $(\mathbb{R}, \mathscr{M}_1, m)$ being the measure-space structure on its domain. Prove that the function

$$F(x) := \int_{(-\infty, x]} f\, dm, \quad x \in \mathbb{R},$$

is continuous.