MATH 222 : ANALYSIS II – MEASURE & INTEGRATION

SPRING 2023

HOMEWORK 6

Instructor: GAUTAM BHARALI **Assigned: FEBRUARY 28, 2023**

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1. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $\{f_n\}$ and $\{g_n\}$ be sequences of $\mathbb{R}$-valued integrable functions on X. Let $f, g : X \to \mathbb{R}$ be two integrable functions, and assume that

$$f_n \longrightarrow f \text{ a.e.}, \quad g_n \longrightarrow g \text{ a.e.}, \quad \text{and}$$
$$|f_n| \leq g_n \quad \forall n \in \mathbb{Z}_+.$$

Also assume that

$$\lim_{n\to\infty} \int_X g_n \, d\mu = \int_X g \, d\mu.$$

Show that the limit on the left-hand side below exists and

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

2. Let $\{(X_\alpha, \mathcal{F}_\alpha)\}_{\alpha\in A}$ be an indexed family of measurable spaces. Assume that A is at most countable. Show that: $\otimes_{\alpha\in A}\mathcal{F}_\alpha$ is generated by the collection

$$\Big\{\prod\nolimits_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{F}_\alpha, \ \alpha \in A\Big\}.$$

3. Let $\{(X_\alpha, \mathcal{F}_\alpha)\}_{\alpha\in A}$ and A be as in Problem 2. Show that if, for each $\alpha \in A$, $\mathcal{F}_\alpha$ is generated by $\mathscr{C}_\alpha \subset \mathscr{P}(X_\alpha)$ with the property that $X_\alpha \in \mathscr{C}_\alpha$, then $\otimes_{\alpha\in A}\mathcal{F}_\alpha$ is generated by the collection

$$\Big\{\prod\nolimits_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{C}_\alpha, \ \alpha \in A\Big\}.$$

4. Let $E \in \mathscr{M}_1$ and let $f : E \to [0, +\infty)$ be a non-negative Lebesgue-measurable function. Show that the set

$$S := \{(x, y) \in E \times \mathbb{R} : 0 \leq y \leq f(x), \ x \in E\}$$

belongs to $\mathscr{M}_1 \otimes \mathscr{M}_1$.