MATH 222 : ANALYSIS II – MEASURE & INTEGRATION
SPRING 2023
HOMEWORK 7

Instructor: GAUTAM BHARALI **Assigned: MARCH 7, 2023**

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1. Let (X_i, d_i), $i = 1, \dots, n$, be metric spaces. Define the *product metric* D on $\prod_{i=1}^n X_i$ as

$$D\big((x_1, \dots, x_n), (y_1, \dots, y_n)\big) := \max_{1 \leq i \leq n} d_i(x_i, y_i) \quad \forall (x_1, \dots, x_n), (y_1, \dots, y_n) \in \prod\nolimits_{i=1}^n X_i.$$

(a) Assume that each X_i contains a countable dense set $\Delta_i \subseteq X_i$, $i = 1, \dots, n$. Show that

$$\otimes_{i=1}^n \mathscr{B}(X_i) = \mathscr{B}\left(\prod\nolimits_{i=1}^n X_i\right).$$

You may freely assume **without proof** that D is a metric.

(b) Now conclude that $\mathscr{B}(\mathbb{R})^{\otimes n} = \mathscr{B}(\mathbb{R}^n)$.

2. Let X be a non-empty set and let $\mathcal{A} \subset \mathscr{P}(X)$ be an algebra. Let

$$\mathscr{C}(\mathcal{A}) := \text{the monotone class generated by } \mathcal{A}.$$

Show that $\mathscr{C}(\mathcal{A}) = \mathcal{F}(\mathcal{A})$.

3. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. Let $f : X \to \mathbb{R}$ be $\mathcal{M}$-measurable and $g : Y \to \mathbb{R}$ be $\mathcal{N}$-measurable. Define $h(x, y) := f(x)g(y)$. Is h $\mathcal{M} \otimes \mathcal{N}$-measurable?

4. Let $-\infty < a < b < +\infty$, write $I := [a, b]$, an interval in $\mathbb{R}$, and let $\phi : I \to \mathbb{R}$ be a continuous, non-negative, strictly increasing function such that $\phi(a) = 0$. Let m denote the Lebesgue measure on $\mathbb{R}$. Define:

$$S := \{(x, y) \in I \times \mathbb{R} : 0 \leq y \leq \phi(x),\ x \in I\}.$$

Let $f : S \to \mathbb{R}$ be in $\mathscr{L}^1\big((m \times m)|_S\big)$. State and prove the intermediate assertions needed to make sense of the following statement:

$$\begin{aligned}\int_S f \, d(m \times m) &= \int_I \left[\int_{[0,\, \phi(x)]} f(x, y) \, dm(y)\right] dm(x) \\ &= \int_{\phi(I)} \left[\int_{[\phi^{-1}(y),\, b]} f(x, y) \, dm(x)\right] dm(y).\end{aligned}$$

Then, prove the above equalities.

Clarification: Do **not** attempt a solution beginning with an auxiliary statement involving simple functions, etc., etc.! Using the conclusions of problems stated in previous assignments, **if** necessary, try to reduce the problem to a suitable application of the Tonelli / Fubini Theorem.

5. Let $(X, \mathcal{F}, \mu)$ be a measure space. With $\mathbb{F}$ denoting either $\mathbb{R}$ or $\mathbb{C}$, show that $\mathscr{L}^\infty(\mu, \mathbb{F})$ is a vector space over $\mathbb{F}$.