MATH 222 : ANALYSIS II – MEASURE & INTEGRATION
SPRING 2023
HOMEWORK 8

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Note: In what follows, if $(X, \mathcal{F}, \mu)$ is a measure space, then $\mathbb{L}^p(\mu)$ — without mention of the underlying field — will denote the $\mathbb{L}^p$-space arising from $\mathbb{R}$-valued measurable functions.

1. (**Minkowski's Integral Inequality**) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and let f be a $\mathcal{M} \otimes \mathcal{N}$-measurable $[0, +\infty)$-valued function on $X \times Y$. Let $1 \leq p < \infty$. Show that

$$\left[\int_X \left|\int_Y f(x,y)\, d\nu(y)\right|^p d\mu(x)\right]^{1/p} \leq \int_Y \left[\int_X f(x,y)^p d\mu(x)\right]^{1/p} d\nu(y).$$

2. Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $0 < p < r \leq \infty$. Show, using Hölder's inequality **appropriately** (note that it is possible that $0 < p < 1$), that $\mathbb{L}^r(\mu, \mathbb{F}) \subset \mathbb{L}^p(\mu, \mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

3. Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $0 < p < r \leq \infty$. Show, **without** using Hölder's inequality, that $\mathbb{L}^r(X, \mu, \mathbb{F}) \subset \mathbb{L}^p(X, \mu, \mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

4. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $0 < p < r < \infty$. Show that $\mathbb{L}^r(\mu, \mathbb{F}) \not\subset \mathbb{L}^p(\mu, \mathbb{F})$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) if and only if, for each $N \in \mathbb{Z}_+$, there exists a set $E_N \in \mathcal{F}$ such that $N < \mu(E_N) < \infty$.

5. Let m_n denote the Lebesgue measure on $\mathbb{R}^n$. Show that $\mathbb{L}^\infty(m_n)$ is not separable.

6. Let $(X, \mathcal{F}, \mu)$ be a measure space. Show that $\mathbb{L}^\infty(\mu, \mathbb{F})$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) is a Banach space.

7. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $1 \leq p < \infty$. Let f_n, f, g_n, g be **real**-valued measurable functions such that

- $g_n \longrightarrow g$ a.e. and $\|g_n\|_\infty \leq M < \infty \ \forall n \in \mathbb{Z}_+$.
- $\{f_n\}$ is a sequence in $\mathbb{L}^p(\mu)$ and $f_n \longrightarrow f$ in $\mathbb{L}^p$-norm.

Show that $f_n g_n \longrightarrow fg$ in $\mathbb{L}^p$-norm.

8. Let $(X, \mathcal{F}, \mu)$ be a measure space. A set $E \in \mathcal{F}$ is called an **atom** if $\mu(E) > 0$ and contains no measurable subset of strictly smaller but positive measure. We say that $(X, \mathcal{F}, \mu)$ is **non-atomic** if there are no atoms in $\mathcal{F}$.

Let $(X, \mathcal{F}, \mu)$ be a non-atomic and σ-finite measure space (where, to avoid trivialities, we take $\mu(X) > 0$). Show that the conclusion of Minkowski's inequality is false on $\mathbb{L}^p(\mu)$ for each $p \in (0, 1)$.