MATH 222 : ANALYSIS II – MEASURE & INTEGRATION
SPRING 2023
HOMEWORK 9

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Note: In what follows, $\mathbb{F}$ will denote either the field $\mathbb{R}$ or the field $\mathbb{C}$.

1. Give an example of a measure space $(X, \mathcal{F}, \mu)$, a sequence $\{f_n\} \subset \mathbb{L}^p(\mu, \mathbb{R})$ and a function $f \in \mathbb{L}^p(\mu, \mathbb{R})$, for some $p \in [1, \infty)$, such that $f_n \longrightarrow f$ in $\mathbb{L}^p$-norm but f_n does not converge to f a.e.

2. (Chebyshev's inequality) Let $(X, \mathcal{F}, \mu)$ be a measure space. Suppose $f : X \to [0, +\infty]$ and that f is integrable. Let $\alpha > 0$. Show that

$$\mu[\{f > \alpha\}] \;\leq\; \alpha^{-1} \int_X f \, d\mu.$$

3. Prove the following:

Theorem. *Let m denote the Lebesgue measure on $\mathbb{R}^n$, $n \in \mathbb{Z}_+$. Then*

(i) If $f : \mathbb{R}^n \to \mathbb{F}$ is Lebesgue measurable, then so is $f(\boldsymbol{\cdot} + y)$ for any $y \in \mathbb{R}^n$.

(ii) If $f : \mathbb{R}^n \to \mathbb{F}$ is Lebesgue integrable, then so is $f(\boldsymbol{\cdot} + y)$ and

$$\int_{\mathbb{R}^n} f(x)\, dm(x) \;=\; \int_{\mathbb{R}^n} f(x + y)\, dm(x) \quad \forall y \in \mathbb{R}^n.$$

(iii) Let $p \in [1, \infty]$. If $f \in \mathbb{L}^p(m, \mathbb{F})$, then so is $f(\boldsymbol{\cdot} + y)$ for any $y \in \mathbb{R}^n$.

Note. For part (ii), first establish this statement for $f = \chi_A$, where $A \in \mathscr{M}_n$. Then, extend the latter conclusion to the general case by following the 3-step construction of the Lebesgue integral and appealing to the linearity of the integral.

4. Let m denote the Lebesgue measure on $\mathbb{R}^n$, $n \in \mathbb{Z}_+$. Fix $y \in \mathbb{R}^n$ and $p \in [1, \infty]$. Define the map $\tau_y : \mathbb{L}^p(m, \mathbb{F})$ as follows:

$$\tau_y(f) \;:=\; f(\boldsymbol{\cdot} + y)$$

(that τ_y maps into $\mathbb{L}^p(m, \mathbb{F})$ follows from the last problem). Show that τ_y is distance-preserving. (Consequently, τ_y is continuous.)

5. Let $(X_i, \|\boldsymbol{\cdot}\|^{(i)})$, $i = 1, 2$, be two normed linear spaces over $\mathbb{F}$. Show that a linear transformation $T : X_1 \to X_2$ is continuous if and only if there exists a constant $C > 0$ such that

$$\sup_{x \,:\, \|x\|^{(1)}} \|T(x)\|^{(2)} \;\leq\; C.$$

6. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of $\mathbb{F}$-valued measurable functions, and let f be a measurable $\mathbb{F}$-valued function. Show that if $f_n \xrightarrow{\mu} f$, then $\{f_n\}$ is Cauchy in measure.

7. Let $(X, \mathcal{F})$ be a measurable space and let μ_1, μ_2 be two measures on $\mathcal{F}$. Suppost **at least one** of $\{\mu_1, \mu_2\}$ is a finite measure. Show that the set function defined by

$$\nu(E) := \mu_1(E) - \mu_2(E) \;\; \forall E \in \mathcal{F}$$

is a signed measure.

8. Fix a function $F : \mathbb{R} \to \mathbb{R}$ that is increasing (not necessarily strictly) and continuous from the **right.** Recall, from Problem 7 of Homework 1, the **outer measure** on $\mathbb{R}$ given by

$$\mu_F^*(A) := \inf \Big\{ \sum_{j \in J} (F(b_j) - F(a_j)) : \{(a_j, b_j] : j \in J\} \text{ is an admissible cover of } A \Big\}.$$

where A is any subset of $\mathbb{R}$. Let $M(\mu_F^*)$ be the σ-algebra associated with μ_F^* given by the Carathéodory construction. We can show that $\mu_F^*((a, b]) = F(b) - F(a)$; **assuming this** if necessary, show that $\mathscr{B}(\mathbb{R}) \subset M(\mu_F^*)$.