MATH 224 : COMPLEX ANALYSIS

SPRING 2016

HOMEWORK 1

Instructor: GAUTAM BHARALI **Assigned: JANUARY 15, 2016**

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1. Let Ω be an open subset of $\mathbb{C}$ and let $f, g : \Omega \longrightarrow \mathbb{C}$. Let f and g be $\mathbb{C}$-differentiable at $a \in \Omega$. **Assuming the product rule** show that if $g(a) \neq 0$, then:

a) (f/g) is $\mathbb{C}$-differentiable at a; and

b) $(f/g)'(a)$ has the expression

$$\left(\frac{f}{g}\right)'(a) \;=\; \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.$$

2. Let $\{a_n\}$ be a series of real numbers. Define

$$A_k := \inf\{a_k, a_{k+1}, a_{k+2}, \dots\},$$
$$B_k := \sup\{a_k, a_{k+1}, a_{k+2}, \dots\}.$$

Show that

$$\liminf_{n\to\infty} a_n = \lim_{k\to\infty} A_k, \quad \text{and} \quad \limsup_{n\to\infty} a_n = \lim_{k\to\infty} B_k.$$

3. Let Ω be a connected open subset of $\mathbb{C}$ and let $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic on Ω. Suppose $f(\Omega)$ is contained in a line in $\mathbb{C}$ that passes through the origin. Show that f is a constant.

Note. You may use, without proof, the following:

Proposition. *Let Ω be a connected open subset of $\mathbb{R}^2$ and $u : \Omega \longrightarrow \mathbb{R}$ a function for which $\partial_x u$ and $\partial_y u$ exist at each point and are continuous on Ω. If $\partial_x u(x,y) = \partial_y u(x,y) = 0$ for each $(x,y) \in \Omega$, then u is a constant.*

4. Show that the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$$

equals 1. Also discuss the convergence of this series at $z = 1, -1$, and i.

5. Derive the identities

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \quad \text{and} \quad \cos(z) = \frac{e^{iz} + e^{-iz}}{2},$$

for every $z \in \mathbb{C}$, giving complete **justifications.**

6–8. Problems 12, 15 and 19 from the exercises to III–Secn. 2 of Conway.