MATH 224 : COMPLEX ANALYSIS
SPRING 2016
HOMEWORK 2

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1. Let $\{a_n\}$ and $\{b_n\}$ be two real sequences and assume that

- $b_n > 0$ for $n = 1, 2, 3, \dots$;
- $\{b_n\}$ converges and $\lim_{n\to\infty} b_n > 0$.

Show that

$$\limsup_{n\to\infty} a_n b_n = \big(\lim_{n\to\infty} b_n\big)\big(\limsup_{n\to\infty} a_n\big).$$

Does this remain true if the positivity requirement on $\lim_{n\to\infty} b_n$ is dropped?

2. Complete the outline below to prove the following:

Theorem. *Let Ω be a connected open set and let $f \in \mathcal{O}(\Omega)$. If $f' \equiv 0$, then f is a constant function.*

Step 1. Fix a point $a \in \Omega$ and set $S := \{z \in \Omega : f(z) = f(a)\}$. Argue that S is closed.

Step 2. Let $w \in S$. We will show that there exists a number $r_w > 0$ such that $D(z_0; r_w) \subset S$. Assume this is **false** and argue that there will be a point $w^* \in \Omega$ such that the closed line segment — denote it by $[w, w^*]$ — is contained in Ω.

Step 3. Consider the function $\Phi : [0,1] \longrightarrow \mathbb{C}$ given by (**note:** the function with which f is composed below is a parametrization of $[w, w^*]$ by the unit interval)

$$\Phi(t) := f((1-t)w + tw^*).$$

Study this function and obtain a contradiction.

Step 4. Now complete the proof.

3. Let Ω be an open subset of $\mathbb{C}$ and let $f \in \mathcal{O}(\Omega)$. Let z_0 be a point in Ω at which $f'(z_0) \neq 0$. Using any relevant result that you know **about $\mathbb{R}^n$-valued maps,** show that there is a neighbourhood $U \subset \Omega$ of z_0 on which f is injective and $V := f(U)$ is an open subset of $\mathbb{C}$. Is the (local) inverse $(f|_U)^{-1}$ holomorphic on V?

4. Let $a, b \in \mathbb{R} \setminus \mathbb{Z}$. Find some condition on a and b that ensures that there is a continuous branch of $z^a(1-z)^b$ defined on $\mathbb{C} \setminus [0,1]$. Then prove that any such branch is holomorphic on $\mathbb{C} \setminus [0,1]$.

5. Let $z_0 = a + ib \in \mathbb{C}$, $a, b \in \mathbb{R}$. Write down formulas, in terms of a and b, of the square roots of z_0.

6–11. Problems 10, 13, 20 and 21 from the exercises to III–Secn. 2, and problems 8 and 9 from the exercises to IV–Secn. 1 of Conway.