MATH 224 : COMPLEX ANALYSIS
SPRING 2016
HOMEWORK 8

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1. Choosing appropriate piecewise-smooth closed curves in $\mathbb{C}$, work out the **real** integrals given in problems 2(c), 2(d), 2(e) and 2(g) from the exercises to V–Secn. 2 of Conway.

2–3. Problems 6 and 7 from the exercises to V–Secn. 2 of Conway.

In what follows, $\widehat{\mathbb{C}}$ will denote the one-point compactification of $\mathbb{C}$. Then:

- Recall that the sets $\big(\mathsf{Ann}(0; R, +\infty) \cup \{\infty\}\big)$ are basic open sets in the topology of $\widehat{\mathbb{C}}$.
- For a subset $S \subset \widehat{\mathbb{C}}$, $\overline{S}^{\infty}$ denotes the closure of S in the topology of $\widehat{\mathbb{C}}$.
- For a subset $S \subset \widehat{\mathbb{C}}$, $\partial_\infty S$ denotes the boundary of S in the topology of $\widehat{\mathbb{C}}$, i.e.,

$$\partial_\infty S := \overline{S}^{\infty} \setminus S.$$

4. Let $\phi : \Omega \longrightarrow \mathbb{R}$ and let $p \in \overline{\Omega}^{\infty}$. Define

$$\limsup_{z \to p} \phi(z) := \lim_{r \to 0^+} \sup\{\phi(z) : z \in \Omega \cap U(p; r)\}$$

where

$$U(p; r) := \begin{cases} D(p; r)^*, & \text{if } p \in \mathbb{C}, \\ \mathsf{Ann}(0; 1/r, +\infty), & \text{if } p = \infty. \end{cases}$$

Show that the limit on the right-hand side of the definition exists (in the **extended** real-line).

Remark. The above definition makes a small correction to the definition given in Chapter VI of Conway (and a **major** correction to that in the 1st Edition, where ϕ occurs as complex-valued).

5. Complete the outline below to prove the following:

Maximum Modulus Theorem. *Let Ω be a domain in $\mathbb{C}$ and $f \in \mathcal{O}(\Omega)$. Suppose there there exists a number $M > 0$ such that $\limsup_{z \to p} |f(z)| \leq M$ for every $p \in \partial_\infty \Omega$. Then, $|f| \leq M$.*

Step 1. Fix a $\delta > 0$. Show that there is an open set $V(\delta)$ — open in the topology of $\widehat{\mathbb{C}}$ — such that $\partial_\infty \Omega \subset V(\delta)$ and $|f(z)| < M + \delta$ for every $z \in \Omega \cap V(\delta)$.

Step 2. Now define

$$H_\delta := \{z \in \Omega : |f(z)| > M + \delta\}.$$

Prove that H_δ is open and bounded.

Step 3. Assume that $H_\delta \neq \varnothing$ and obtain a contradiction.

Step 4. Now complete the proof.

6. Let p be a non-constant polynomial, let $n = \deg(p)$, and let $R > 0$.

a) Show that each connected component of $\{z \in \mathbb{C} : |p(z)| < R\}$ is bounded.

b) Show that the set $\{z \in \mathbb{C} : |p(z)| < R\}$ cannot have more than n connected components.

7. Problem 2 from the exercises to VI–Secn. 2 of Conway.