MA 329 : TOPICS IN SEVERAL COMPLEX VARIABLES
AUTUMN 2022
HOMEWORK 1

Instructor: GAUTAM BHARALI **DUE: Friday, Sep. 9, 2022**

Note:

a) You are allowed to discuss these problems with your classmates, but individually-written and **original** write-ups are expected for submission. Please **acknowledge** any persons from whom you received help in solving these problems.

b) Given a multi-index $\alpha \in \mathbb{N}^n$, we shall use the following notation:

$$\begin{aligned} |\alpha| &:= \alpha_1 + \dots + \alpha_n \quad \text{and} \quad \alpha! := \alpha_1! \dots \alpha_n!, \\ z^\alpha &:= z_1^{\alpha_1} \dots z_n^{\alpha_n}, \\ \frac{\partial^{|\alpha|}}{\partial z^\alpha} &:= \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}. \end{aligned}$$

c) If f is a function or a map on an open set $\Omega \subseteq \mathbb{C}^n$ and f is $\mathbb{C}$-differentiable at $z_0 \in \Omega$, then we shall denote the complex total derivative of f at z_0 as $Df(z_0)$.

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1. Let Ω be an open set in $\mathbb{C}^n$, $z_0 \in \Omega$, and $f, g : \Omega \to \mathbb{C}$ be $\mathbb{C}$-differentiable at z_0. Show that fg is also $\mathbb{C}$-differentiable at z_0 and that

$$D(fg)(z_0) := f(z_0)Df(z_0) + g(z_0)Dg(z_0).$$

2. Let $X_1, \dots X_n \in \mathbb{C}$ and suppose $|X_j| < 1$ for $j = 1, \dots, n$. Show that $\sum_{\alpha \in \mathbb{N}^n} X^\alpha$ is absolutely convergent and that

$$\sum_{\alpha \in \mathbb{N}^n} X^\alpha = \prod_{j=1}^n \Big(\sum_{\nu \in \mathbb{N}} X_j^\nu \Big).$$

Hint. Although this is **not** the only approach to the solution, consider defining an auxiliary function that you know is holomorphic on $\mathbb{D}^n$ (where $\mathbb{D} := D(0, 1) \subset \mathbb{C}$).

3. Prove the following result:
Let Ω be a domain in $\mathbb{C}^n$, $n \geq 2$, and let $f : \Omega \to \mathbb{C}$ be holomorphic. The set $f^{-1}\{0\}$ contains no isolated points.
using the 1-variable Hurwitz Theorem.

4. (*Cauchy's estimates*) Let Ω be an open subset of $\mathbb{C}^n$ and let $f : \Omega \to \mathbb{C}$ be holomorphic. Let $a \in \Omega$. Let $R > 0$ be such that $\overline{\Delta(a; (R, \dots, R))} \subset \Omega$. Show that there exist constants $C_\alpha > 0$ such that

$$\left| \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(a) \right| \leq \frac{C_\alpha}{R^{|\alpha|}} \sup_{w : |w_j - a_j| = R,\, 1 \leq j \leq n} |f(w)|,$$

where the constants C_α are **independent** of a, R, and f.
Tip. You may use without proof the uniqueness of coefficients of a power-series development of a holomorphic function.

5. State Liouville's theorem for $\mathbb{C}^n$, $n \geq 2$, and give a proof of it:

- firstly, using Cauchy's estimates for $n \geq 2$;
- next, **without** using Cauchy's estimates for $n \geq 2$.

6. State the Maximum Modulus Theorem for $\mathbb{C}^n$, $n \geq 2$, and prove it using the Open Mapping Theorem.

7. Let Ω be an open subset of $\mathbb{C}^n$ and let $F : \Omega \to \mathbb{C}^m$, $m \geq 2$. Write $F = (f_1, \dots, f_m)$. Show that F is holomorphic (i.e., that it is $\mathbb{C}$-differentiable) if an only if each f_j, $j = 1, \dots, m$, is a holomorphic function.

8. Let $\Omega_j \subseteq \mathbb{C}^{n_j}$ be domains, $j = 1, 2$. We had constructed in class a metric ρ on $\mathcal{C}(\Omega_1, \Omega_2)$. Show that the metric topology generated by ρ coincides with the topology of local uniform convergence on $\mathcal{C}(\Omega_1, \Omega_2)$.

9. Let Ω be a domain in $\mathbb{C}^n$, let f be holomorphic on Ω and suppose $f \not\equiv 0$. Show that $\Omega \setminus f^{-1}\{0\}$ is connected.

10. Given two domains $\Omega, \Omega' \subseteq \mathbb{C}^n$, a holomorphic map $F : \Omega \to \Omega'$ is called a *biholomorphism* if F is bijective and F^{-1} is also holomorphic. An *automorphism of* Ω is a biholomorphism of Ω onto itself. Let G be a bounded domain in $\mathbb{C}$. Give a description — e.g., in terms of functions on G or in any other manner that provides more information — of the automorphisms of $(G \times \mathbb{C})$.