MA 329 : TOPICS IN SEVERAL COMPLEX VARIABLES
AUTUMN 2022
HOMEWORK 2

Instructor: GAUTAM BHARALI **DUE: Friday, Oct. 28, 2022**

Note:

a) You are allowed to discuss these problems with your classmates, but individually-written and **original** write-ups are expected for submission. Please **acknowledge** any persons from whom you received help in solving these problems.

b) Given a multi-index $\alpha \in \mathbb{N}^n$, we shall use the following notation:

$$|\alpha| := \alpha_1 + \dots + \alpha_n \quad \text{and} \quad \alpha! := \alpha_1! \dots \alpha_n!,$$
$$z^\alpha := z_1^{\alpha_1} \dots z_n^{\alpha_n},$$
$$\frac{\partial^{|\alpha|}}{\partial z^\alpha} := \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}.$$

c) If f is a function or a map on an open set $\Omega \subseteq \mathbb{C}^n$ and f is $\mathbb{C}$-differentiable at $z_0 \in \Omega$, then we shall denote the complex total derivative of f at z_0 as $Df(z_0)$.

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1. Let $\Omega \subseteq \mathbb{C}^n$ be a domain and V an analytic subvariety of Ω. Show that if $V \subsetneq \Omega$, then $\Omega \setminus V$ is dense in Ω.

2. Let $\Omega \subseteq \mathbb{C}^n$ be a domain and let $\varphi_1, \dots \varphi_m \in \mathcal{O}(\Omega)$. Suppose $\sum_{j=1}^m |\varphi_j|^2$ is a constant function. Show that each φ_j, $j = 1, \dots, m$, is constant.

3. Complete the following outline to prove that if $\Omega \subseteq_{\text{open}} \mathbb{C}^n$, then any compact analytic subvariety of Ω is a finite set.

(*a*) Prove the above result for the case $n = 1$.

(*b*) Assuming that the above result is true for $n = m - 1$ for some $m \in \mathbb{Z}_+ \setminus \{1\}$, prove the corresponding result for $n = m$ by using the following theorem:

Projection Theorem. *Let $D \subseteq_{\text{open}} \mathbb{C}^m$, $m \geq 2$, and let V be an analytic subvariety of D. Let $p \in V$. Write*

$$\mathscr{L} := \{\pi_{m-1}(p)\} \times \mathbb{C},$$

where π_{m-1} denotes the projection onto the first $(m-1)$ coordinates. If p is an isolated point of $V \cap \mathscr{L}$, then there exists an m-dimensional polydisc Δ with centre p such that $\Delta \subset D$ and such that $\pi_{m-1}(V \cap \Delta)$ is an analytic subvariety of $\pi_{m-1}(\Delta)$.

You may freely use the above theorem **without** proving it.

4. Following the idea of the proof of the Remmert–Stein Theorem B for the $n = 2$ case, and stating and proving whatever else is required, prove the Remmert–Stein Theorem B in its full generality.

5. Let $f : \mathbb{D} \to \mathbb{D}$ be proper and holomorphic. Show that there exist points $a_1, \dots, a_m \in \mathbb{D}$, positive integers $n_1, \dots, n_m$, and a constant $\theta \in \mathbb{R}$ such that

$$f(z) = e^{i\theta} \prod_{j=1}^m \left(\frac{z - a_j}{1 - \overline{a}_j z} \right)^{n_j}, \quad z \in \mathbb{D}.$$

6. Show that

$$\mathcal{M}_{\mathbb{D}}(a,b) := \left|\frac{a-b}{1-\overline{a}b}\right| \quad \forall a,b \in \mathbb{D}$$

is a metric on $\mathbb{D}$.

6. Let p denote the Poincaré distance on $\mathbb{D}$. Show that if $b \in (0,1)$, then the p-segment $[0,b]_{\mathsf{p}} = [0,b]$.

7. Let $D \subseteq \mathbb{C}^n$ be a domain, let V be an analytic subvariety of D such that $V \subsetneq D$. Let $\Omega := D \setminus V$. Find a relation between $c_D(a,b)$ and $c_\Omega(a,b)$ for all $a,b \in \Omega$. Here, $c_\star$ denotes the Carathéodory distance.

8. Let Ω denote a domain in $\mathbb{C}^n$ and assume that it is Carathéodory hyperbolic. Is the metric space (Ω, c_Ω) necessarily Cauchy-complete? Please **justify** your answer.