MA 329 : TOPICS IN SEVERAL COMPLEX VARIABLES

AUTUMN 2022

HOMEWORK 3

Instructor: GAUTAM BHARALI **DUE: Wednesday, Nov. 23, 2022**

Note:

a) You are allowed to discuss these problems with your classmates, but individually-written and **original** write-ups are expected for submission. Please **acknowledge** any persons from whom you received help in solving these problems.

b) Given a multi-index $\alpha \in \mathbb{N}^n$, we shall use the following notation:

$$\begin{aligned} |\alpha| &:= \alpha_1 + \dots + \alpha_n \quad \text{and} \quad \alpha! := \alpha_1! \dots \alpha_n!, \\ z^\alpha &:= z_1^{\alpha_1} \dots z_n^{\alpha_n}, \\ \frac{\partial^{|\alpha|}}{\partial z^\alpha} &:= \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}. \end{aligned}$$

c) If f is a function or a map on an open set $\Omega \subseteq \mathbb{C}^n$ and f is $\mathbb{C}$-differentiable at $z_0 \in \Omega$, then we shall denote the complex total derivative of f at z_0 as $Df(z_0)$.

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1. Let Ω be a domain in $\mathbb{C}^n$. Show that the Kobayashi pseudometric K_Ω satisfies the triangle inequality.

2. Let a be a point in the open unit (Euclidean) ball $\mathbb{B}^n \subset \mathbb{C}^n$ and recall the map

$$\Psi_a(z) := \frac{a - P_a(z) - \sqrt{1 - \|a\|^2}\, Q_a(z)}{1 - \langle z, a \rangle}, \quad \forall z \in \mathbb{B}^n,$$

where P_a is the orthogonal projection onto $\text{span}_\mathbb{C}\{a\}$, Q_a is the orthogonal projection onto $\big(\text{span}_\mathbb{C}\{a\}\big)^\perp$, and $\langle \cdot\,, \cdot \rangle$ is the standard Hermitian inner product on $\mathbb{C}^n$. Show that

$$1 - \langle \Psi_a(z), \Psi_a(w) \rangle = \frac{(1 - \|a\|^2)(1 - \langle z, w \rangle)}{(1 - \langle z, a \rangle)(1 - \langle a, w \rangle)} \quad \forall z, w \in \mathbb{B}^n.$$

Using the above, show that

(*a*) $\Psi_a(\mathbb{B}^n) \subseteq \mathbb{B}^n$.

(*b*) $\Psi_a \circ \Psi_a = \mathsf{id}_{\mathbb{B}^n}$.

3. Let Ω be a Kobayashi hyperbolic domain. Prove, relying on first principles/the definition of K_Ω that K_Ω-balls are connected relative to the standard topology on Ω.

4. Let $\widehat{\mathbb{C}}$ denote the Riemann sphere equipped with the complex structure represented by the atlas $\{(\mathbb{C}, \mathsf{id}_\mathbb{C}), ((\mathbb{C}^* \cup \{\infty\}), \mathsf{inv})\}$, where

$$\mathsf{inv}(z) := \begin{cases} 1/z, & \text{if } z \in \mathbb{C}^*, \\ 0, & \text{if } z = \infty. \end{cases}$$

Show that $\widehat{\mathbb{C}}$ is biholomorphic to $\mathbb{CP}^1$.

5. Let ω_1 and ω_2 be two non-zero complex numbers that are $\mathbb{R}$-independent when viewed as vectors in $\mathbb{R}^2$. Let us view the (real) 2-dimensional torus $\mathbb{T}^2$ as $\mathbb{T}^2 = S^1 \times S^1$ (equipped with the relative topology that it inherits from $\mathbb{C} \times \mathbb{C}$). Let us write

$$U_{00} := \{(e^{i\theta_1}, e^{i\theta_2}) \in \mathbb{T}^2 : 0 < \theta_1, \theta_2 < 2\pi\},$$
$$\phi_{00} : U_{00} \ni (e^{i\theta_1}, e^{i\theta_2}) \longmapsto \frac{\theta_1}{2\pi}\omega_1 + \frac{\theta_2}{2\pi}\omega_2.$$

You may **assume without proof** that U_{00} is open in $\mathbb{T}^2$ and that ϕ_{00} is a homeomorphism onto its image in $\mathbb{C}$. Emulating the above idea, and the fact that the same point in $\mathbb{T}^2 \hookrightarrow \mathbb{C}^2$ can be represented by

$$\big(e^{i(\theta_1+2\pi\mu)}, e^{i(\theta_2+2\pi\nu)}\big) \text{ for some } \theta_1, \theta_2 \in [0, 2\pi) \text{ and } \forall(\mu, \nu) \in \mathbb{Z}^2,$$

construct **four** charts (one of which is given above)

$$(U_{jk}, \phi_{jk}), \quad \phi_{jk} : U_{jk} \to \mathbb{C}, \quad j = 0, 1, \ k = 0, 1,$$

that cover $\mathbb{T}^2$ such that

$$\mathscr{A} := \{(U_{ij}, \phi_{jk}) : j = 0, 1, \ k = 0, 1\}$$

is a **holomorphic** atlas on $\mathbb{T}^2$.

6. Let X and Y be connected complex manifolds and let $\pi : Y \to X$ be a holomorphic covering map. Then, show that

$$K_X(x_1, x_2) = \inf\{K_Y(\widehat{x}_1, \widehat{x}_2) : \widehat{x}_j \in \pi^{-1}\{x_j\}, \ j = 1, 2\}.$$

7. *This problem is meant to establish an analytical characterization of convexity of a domain whose boundary is an embedded $\mathcal{C}^2$-submanifold.* To this end, consider a domain $\Omega \subsetneq \mathbb{R}^N$, $N \geq 2$, with $\partial\Omega$ an embedded $\mathcal{C}^2$-smooth submanifold of $\mathbb{R}^N$, and provide details for the following outline. **Fix** a defining function ϱ of class $\mathcal{C}^2$. Pick an (arbitrary) $p \in \partial\Omega$.

(*a*) Describe an explicit affine change of coordinate $\tau : \mathbb{R}^N \to \mathbb{R}^N$ such that $\tau(p) = 0$ and, using the notation
$$(y_1, \dots, y_N) := \tau(x_1, \dots, x_N),$$
we have $d\tau(p)\big(\nabla\varrho(p)\big) = \|\nabla\varrho(p)\|(0, \dots, 0, -1)$ and $d\tau(p) : T_p(\partial\Omega) \to_{\text{onto}} \{y \in \mathbb{R}^N : y_N = 0\}$ (here, $d\tau$ denotes the **real** total derivative of τ).

(*b*) Re-express the equation of $\partial\Omega$ in $(y_1, \dots, y_N)$-coordinates: i.e., show that there exists an open ball B centred at $y = 0$ and a function $\varphi : \big(B \cap \{y : y_N = 0\}\big) \to \mathbb{R}$ with appropriate properties such that
$$\partial(\tau(\Omega)) \cap B = \{y \in B : y_N = \varphi(y_1, \dots, y_{N-1})\}.$$

(*c*) From (*a*) and (*b*), deduce that Ω is convex if and only if
$$\sum_{j,k=1}^{N} \frac{\partial^2\varrho}{\partial x_j \partial x_k}(p) V_j V_k \geq 0 \quad \forall V \in T_p(\partial\Omega) \text{ and } \forall p \in \partial\Omega.$$

Hint. You may use **without** proof the fact that affine maps preserve convexity. Secondly, you may use — although there are other ways to approach part (*c*) — **without** proof a statement about the preservation of inertia of suitable quadratic forms of the type discussed in class.