MATH 380 : INTRODUCTION TO COMPLEX DYNAMICS

AUTUMN 2016

HOMEWORK 1

Instructor: GAUTAM BHARALI **DUE: Monday, Aug. 29, 2016**

Remarks and instructions :

- You are allowed to discuss these problems with your fellow-students, but individually-written and **original** write-ups are expected for submission.

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1. Let X be a locally Euclidean topological space and let $\mathfrak{A}$ be a complex atlas on X that gives it the structure of a Riemann surface. Show that there is a unique complex structure $\widetilde{\mathfrak{A}} \subseteq \mathfrak{A}$.

2. Let X be a Riemann surface and $\mathfrak{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in J}$ a complex atlas on X. Let $\widetilde{\mathfrak{A}}$ be the complex structure containing $\mathfrak{A}$ (refer to Problem 1 to see that there is precisely one such complex structure). Fix $\alpha_0 \in J$ and let $V \subsetneq U_{\alpha_0}$ be a non-empty open set. Show that

$$\big(V, \phi_{\alpha_0}|_V\big) \in \widetilde{\mathfrak{A}}.$$

3. Let X be a locally Euclidean topological space that has the structure of a Riemann surface with respect to two complex structures $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ and $\{(V_\beta, \psi_\beta)\}_{\beta \in J}$. Show that these two structures are equivalent if and only if the identity map $\mathsf{id}_X : (X, \{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}) \longrightarrow (X, \{(V_\beta, \psi_\beta)\}_{\beta \in J})$ is holomorphic.

4. Recall that $\mathbb{CP}^1$ is the quotient space

$$\mathbb{C}^2 \setminus \{(0,0)\} \,/ \sim$$
$$\text{where } (x_0, x_1) \sim (y_0, y_1) \iff (x_0, x_1) = \lambda(x_0, x_1) \text{ for some } \lambda \in \mathbb{C} \setminus \{0\}.$$

Let $[x_0 : x_1]$ denote the equivalence class of (x_0, x_1), and $U_j := \{[x_0 : x_1] \in \mathbb{CP}^1 : x_j \neq 0\}$, $j = 0, 1$.

a) Write $\phi_0 : U_0 \ni [x_0 : x_1] \longmapsto x_1/x_0$. Show that ϕ_0 does not depend on the choice of representative of $[x_0 : x_1]$, and that $\phi_0 : U_0 \longrightarrow \mathbb{C}$ is a homeomorphism.

b) Recall that $\{(U_j, \phi_j) : j = 0, 1\}$ is a complex atlas on $\mathbb{CP}^1$. Show that the Riemann surface $\mathbb{CP}^1$ is biholomorpic to $\widehat{\mathbb{C}}$.

5. (**This problem presents, perhaps, the most computationally direct proof that there exist uncountably many inequivalent complex structures on the torus.**) View the torus $\mathbb{T}^2$ as $S^1 \times S^1 \subset \mathbb{C}^2$ equipped with the relative topology.

a) Let ω_1 and ω_2 be two non-zero complex numbers that that are $\mathbb{R}$-independent when viewed as vectors in $\mathbb{R}^2$. Thus, any $z \in \mathbb{C}$ can be written uniquely as

$$z = \theta_1(z)\omega_1 + \theta_2(z)\omega_2, \quad \theta_1, \theta_2 \in \mathbb{R}.$$

Define $p_\omega(z) := (e^{i\theta_1(z)}, e^{i\theta_2(z)})$. Show that $\mathbb{T}^2$ can be endowed with a complex structure with respect to which $p_\omega : \mathbb{C} \longrightarrow \mathbb{T}^2$ is a holomorphic map.

b) Let $(\omega_1, \omega_2) = (1, \omega)$ and $(\tau_1, \tau_2) = (1, \tau)$ be two $\mathbb{R}$-independent pairs of complex numbers. Based on the construction in (a), endow $\mathbb{T}^2$ with two complex structures; denote them by $\mathfrak{A}_\omega$ and $\mathfrak{A}_\tau$. Give a necessary and sufficient condition for $\mathfrak{A}_\omega$ and $\mathfrak{A}_\tau$ to be equivalent.

6. Let $p : X \longrightarrow Y$ be a holomorphic covering map between Riemann surfaces. Suppose $f : U \longrightarrow Y$ is a holomorphic map, where U is a Riemann surface, such that f lifts to a continuous map $F_f : U \longrightarrow X$. Show that F_f is holomorphic.