MATH 380 : INTRODUCTION TO COMPLEX DYNAMICS

AUTUMN 2016

HOMEWORK 3

Instructor: GAUTAM BHARALI **DUE: Monday, Oct. 24, 2016**

Remarks and instructions :

- You are allowed to discuss these problems with your fellow-students, but individually-written and **original** write-ups are expected for submission.

b) We shall use the following notation:

$$\mathbb{D} := \text{the open unit disc with centre } 0 \in \mathbb{C},$$
$$\widehat{\mathbb{C}} := \text{the one-point compactification of } \mathbb{C}.$$

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1. Let $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be a holomorphic map and assume that $\deg(f) \geq 2$. Let $x_0 \in \widehat{\mathbb{C}}$ have finite grand orbit under f. Using the following result (the local version of which you have studied as the "counting zeros theorem"):

> For a point $w \in \widehat{\mathbb{C}}$, $\mathsf{Card}[f^{-1}(w)] < \deg(f)$ if and only if at least one $z \in f^{-1}\{w\}$ is a critical point of f.

show that each point in $GO_f(x_0)$ is a critical point of f.

Hint. Recall that, if $p \in GO_f(x_0)$, then p belongs to a periodic orbit. You may also use without proof, if required, the fact that $\deg(f^n) = \deg(f)^n$.

2. Let $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be a holomorphic map with $\deg(f) \geq 2$. Show that the Julia set of f is a perfect set.

3. Let $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be a holomorphic map and let z_0 be an attracting fixed point of f. Let $\mathscr{B}_f(z_0)$ denote the basin of attraction of z_0. Show that, for each $p \in \mathscr{B}_f(z_0)$, $GO_f(p) \subset \mathscr{B}_f(z_0)$. Now suppose $GO_f(p)$ is not finite: then, is $GO_f(p)$ discrete in $\mathscr{B}_f(z_0)$?

4. Let $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be a holomorphic map and assume that, in a neighbourhood of 0,

$$f(z) = z + a_{p+1}z^{p+1} + O(|z|^{p+2}),$$

where $p \geq 1$ and $a_{p+1} \neq 0$. Let $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_{2p-1}\}$ be a cyclic ordering, along the circle $\{w \in \mathbb{C} : |w| = 1/|a_{p+1}|^{1/p}\}$, of the attracting and repelling directions of f associated to the fixed point 0, with $\boldsymbol{v}_0$ denoting a repelling direction. Show that there exist constants r_0, $\theta_0 > 0$, sufficiently small, such that

$$|f(z)| \begin{cases} > |z| & \text{on the sector } S_j, \text{ if } j \text{ is even}, \\ < |z| & \text{on the sector } S_j, \text{ if } j \text{ is odd}, \end{cases}$$

where the sectors $S_0, \dots, S_{2p-1}$ are defined as:

$$S_j \;:=\; \{z \in \mathbb{C} : 0 < |z| < r_0,\; |\mathsf{Arg}(z/\boldsymbol{v}_j)| < \theta_0\}.$$

Here, Arg denotes the principal branch of the argument.