Welcome to the inaugural lecture of MA380: Introduction to Complex Dynamics!

Gautam Bharali

Indian Institute of Science

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Course Planning & Rudiments of Dynamics

Online: March 2, 2021

Spring Semester, 2021

Step 0: the MA380 team

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Communications related to MA380 — except for the homework assignments themselves — will posted on MS Teams. These will be push-notifications; you **will be alerted of a(n unread) communication via e-mail.**

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Please respond to the ongoing survey if you haven't done so; **deadline:** 12:00 noon of March 3.

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We often impose the following condition:

(Semigroup condition *OR* autonomy) The state of the process at time $\tau \geq t_0$ depends **only** on the state of the process at t_0 and the time elapsed (i.e., $\tau - t_0$). So:

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$$F(x, t_1 + t_0) = F\big(F(x, t_0), t_1\big) \;\; \forall t_0, t_1 \in I \text{ and } \forall x \in X.$$

The rudiments of dynamical systems, cont'd.

If we write $f_t := F(\cdot, t)$, then

$$f_0 = \mathsf{id}_X, \qquad [\text{by definition of } F]$$

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Also, check that if $I = \mathbb{N}$, then, by mathematical induction:

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Suppose we know there is a **unique** soln. $Y_x \in \mathcal{C}^1([0, +\infty); \mathbb{R}^n)$ of the IVP $(*)_x$. (Check) With $F(x, t) := Y_x(t)$, $(\mathbb{R}^n, F, [0, +\infty))$ is an autonomous dynam. syst.