MATH 380 : INTRODUCTION TO COMPLEX DYNAMICS

SPRING 2021

HOMEWORK 2

Instructor: GAUTAM BHARALI | **DUE: ~~Saturday, April 10, 2021~~ Sunday, April 11, 2021**

Remarks and instructions:

a) You are allowed to discuss these problems with your fellow-students, but individually-written and **original** write-ups are expected for submission.

b) Please **acknowledge** any persons from whom you received help in solving these problems — stating the problem(s) in which you took their help.

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1. Show that

$$\mathcal{M}_{\mathbb{D}}(z_1, z_2) := \left| \frac{z_1 - z_2}{1 - \overline{z}_2 z_1} \right| \quad \forall z_1, z_2 \in \mathbb{D}$$

is a metric on $\mathbb{D}$.

2. Let X be a hyperbolic Riemann surface and let d_X denote the Kobayashi distance on X. Show that (X, d_X) is Cauchy complete.

Remarks: You may use without proof any property — whether proved in class or not — of the Poincaré distance $p_{\mathbb{D}}$ stated in class. In most textbooks on Riemann surfaces where the construction of d_X follows a differential-geometric approach, d_X is called the *hyperbolic distance* on X.

3. Prove that $\widehat{\mathbb{C}}$ is the unique (up to biholomorphic equivalence) elliptic Riemann surface.

Tip. If you are unable to prove this without the use of **just** the results presented in class, then you may: (a) assume without proof that Riemann surfaces are oriented surfaces, and (b) appeal to the topological classification of compact orientable surfaces.

4. Let Y be a non-compact hyperbolic Riemann surface. Show that a set $K \subset Y$ is compact if and only if K is closed and is bounded with respect to d_Y.

5. Give a **rigorous** and complete proof that any element $f \in \mathrm{Aut}(\mathbb{C})$ is of the form $f(z) = az + b$, where $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$.

6. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a non-constant holomorphic map, $f \not\equiv \mathsf{id}_{\widehat{\mathbb{C}}}$, having a non-empty Fatou set. Let Ω be a connected component of the Fatou set. Show that $f(\Omega)$ is also a connected component of the Fatou set of f.

7. Given two Riemann surfaces X and Y, recall that the definition of a holomorphic map $f : X \to Y$ (with holomorphic atlases $\mathscr{A}_X$ and $\mathscr{A}_Y$ on X and Y, respectively, implicit in the definition) requires f to be *a priori* continuous. Why do we have this requirement?

Tip. In case you are attempting, with some pair of Riemann surfaces X and Y, to cook up holomorphic atlases $\mathscr{A}_X$ and $\mathscr{A}_Y$ on X and Y, respectively, that are pathological in a way that a **discontinuous** function $f : X \to Y$ satisfies the chart-wise conditions of holomorphicity — even a credible attempt at this requires tools that we haven't studied. Think more fundamentally; the above question is a test of whether you are comfortable with **basic** definitions.

The following problem will go a little beyond what has been taught until now. You will need the material from the **lecture of April 6** for a definition of the words/notation used in it.

8. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map, and let ξ be a fixed point of f. Show the following relationship between the relevant multipliers:

$$\lambda_f(\xi) = \lambda_{\tau \circ f \circ \tau^{-1}}\big(\tau(\xi)\big),$$

where τ is a Möbius transformation.