MATH 380 : INTRODUCTION TO COMPLEX DYNAMICS

SPRING 2021

HOMEWORK 3

Instructor: GAUTAM BHARALI **DUE: Saturday, May 15, 2021**

Remarks and instructions:

a) You are allowed to discuss these problems with your fellow-students, but individually-written and **original** write-ups are expected for submission.

b) Please **acknowledge** any persons from whom you received help in solving these problems — stating the problem(s) in which you took their help.

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1. This problem draws upon:

- the notation and construction in Problem 6 of the first asignment;
- the conclusion of Problem 1 of the first assignment (although there **are** other ways to solve the problem).

Write $\boldsymbol{\omega} := (\omega_1, \omega_2)$, where ω_1 and ω_2 are two non-zero complex numbers that are $\mathbb{R}$-independent when viewed as vectors in $\mathbb{R}^2$. Let

$$\mathscr{A}^{\boldsymbol{\omega}} := \big\{(U^{\boldsymbol{\omega}}_{ij}, \phi^{\boldsymbol{\omega}}_{jk}) : j = 0, 1,\ k = 0, 1\big\}$$

denote the holomorphic atlas on the torus $\mathbb{T}^2 = S^1 \times S^1$ as introduced in Problem 6 of the first assignment. Now write $\boldsymbol{\tau} := (\tau_1, \tau_2)$, where τ_1 and τ_2 are also two non-zero complex numbers that are $\mathbb{R}$-independent. Find a necessary condition on $\boldsymbol{\omega}$ and $\boldsymbol{\tau}$ such that $\mathscr{A}^{\boldsymbol{\omega}} \sim \mathscr{A}^{\boldsymbol{\tau}}$. **Do note:** you can use without proof — but you have to be **correct** — your conclusions from Problem 6 of the first asignment.

Remark: Unless you have stated an overly permissive condition, the condition that you have discovered is also sufficient — but this can be a bit laborious to show.

2. Study Section 4.3 from the book by Beardon (and Appendix II in case you require further information on the Weierstrass $\wp$-function) for an example of a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ whose Julia set is $\widehat{\mathbb{C}}$.

Remark: Notice the connection between the material in Appendix II and Problem 1 above.

3. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map having a repelling periodic orbit. Show that this orbit lies in the Julia set of f.

4. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map and assume that $\deg(f) \geq 2$.

(*a*) Let $z_0 \in \widehat{\mathbb{C}}$ have finite grand orbit under f. Using the following result (the **local** version of which you have studied as the "counting zeros theorem"):

> For a point $w \in \widehat{\mathbb{C}}$, $\mathsf{Card}\big(f^{-1}\{w\}\big) < \deg(f)$ if and only if at least one $z \in f^{-1}\{w\}$ is a critical point of f.

show that each point in $GO_f(z_0)$ is a critical point of f.

(b) Show that $\mathscr{E}_f$ is a union of superattracting periodic orbits.

Hint. You may use without proof, if required, the fact that $\deg(f^n) = \deg(f)^n$.

5. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map that fixes 0 and assume that 0 is a repelling fixed point. Show that there exists a holomorphic map $\varphi : \mathbb{C} \to \widehat{\mathbb{C}}$, satisfying $\varphi(0) = 0$, that is biholomorphic on a small neighbourhood of zero and such that the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{M_\lambda} & \mathbb{C} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \widehat{\mathbb{C}} & \xrightarrow[f]{} & \widehat{\mathbb{C}} \end{array}$$

commutes. Here, $M_\lambda : z \longmapsto \lambda z$, where $\lambda = f'(0)$.

6. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map with $\deg(f) \geq 2$. Show that the Julia set of f is a perfect set.

7. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map and let z_0 be an attracting fixed point of f. Let $\mathscr{B}_f(z_0)$ denote the basin of attraction of z_0. Show that, for each $p \in \mathscr{B}_f(z_0)$, $GO_f(p) \subset \mathscr{B}_f(z_0)$. Now suppose $GO_f(p)$ is not finite: then, is $GO_f(p)$ discrete in $\mathscr{B}_f(z_0)$?

8. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map and let $\deg(f) \geq 2$. Let z_0 be an attracting fixed point of f. Let $\mathscr{B}_f(z_0)$ denote the basin of attraction of z_0. **True or false:** The connected component of $\mathscr{B}_f(z_0)$ containing z_0 contains at least one critical point of f.