MATH 380 : INTRODUCTION TO COMPLEX DYNAMICS

SPRING 2021

HOMEWORK 4

Instructor: GAUTAM BHARALI **DUE: Tuesday, June 1, 2021**

Remarks and instructions:

a) You are allowed to discuss these problems with your fellow-students, but individually-written and **original** write-ups are expected for submission.

b) Please **acknowledge** any persons from whom you received help in solving these problems — stating the problem(s) in which you took their help.

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1. Let ϱ denote a (family of) norm(s) on $\mathbb{R}^d$ with a uniform description for $d = 1, 2, 3, \dots$. Fix $d \in \mathbb{Z}_+$ and let ${}^{\varrho}H_t$ be the outer measure on $\mathbb{R}^d$ defined by replacing $\mathscr{C}_\varepsilon(S)$ by $\mathscr{C}_\varepsilon(S; \varrho)$ in the definition of H_t — the t-dimensional Hausdorff outer measure on $\mathbb{R}^d$. Show that there exists a constant $C \equiv C(d) > 0$ such that

$$H_t(S) \leq {}^{\varrho}H_t(S) \leq C H_t(S) \quad \forall S \subseteq \mathbb{R}^d.$$

2. Fix $d \in \mathbb{Z}_+ \setminus \{1\}$ and write $S_k := \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_j = 0 \ \forall j = k+1, \dots, d\}$, where $k \in \mathbb{Z}_+$ and $k < d$. Show that $\dim_H(S_k) = k$.

3. Let K_∞ be the Cantor set described as follows. Let $K_0 := [0, 1]$. **Fix** $\alpha \in (0, 1)$. For **each** $n = 0, 1, 2, \dots$, we define:

K_{n+1} := the set obtained by removing open intervals that form the middle α^{th} part of each connected component of K_n.

Define

$$K_\infty := \bigcap_{n=1}^{\infty} K_n.$$

Compute the Hausdorff dimension of K_∞.

4. Let (X, d) be a metric space and let $p \in X$. We say that a function $f : X \to \mathbb{R}$ is *lower semicontinuous at* p if

$$\liminf_{x \to p} f(x) \geq f(p),$$

and we say that f is *lower semicontinuous* if it is lower semicontinuous at every $p \in X$.

(*a*) Let U be a non-empty open subset of X. Show that the characteristic function χ_U is lower semicontinuous.

(*b*) Assume the following result (proof not required):

Let f be a non-negative lower-semicontinuous function on $\widehat{\mathbb{C}}$. Then, there exists a monotone-increasing sequence of real-valued continuous functions $\{\varphi_\nu\}$ such that $\lim_{\nu\to\infty}\varphi_\nu(z) = f(z)$ *for each* $z \in \widehat{\mathbb{C}}$.

Let μ be a Borel probability measure on $\widehat{\mathbb{C}}$ and let $\{\mu_n\}$ be a sequence of Borel probability measures on $\widehat{\mathbb{C}}$ such that $\mu_n \longrightarrow \mu$ in the weak* topology. Also assume that the sequence $\{\mu_n\}$ is *uniformly inner regular:* i.e., given any non-empty open set Ω and any $\varepsilon > 0$, there exist $N \equiv N(\Omega, \varepsilon) \in \mathbb{Z}_+$ and a compact $K_\varepsilon \subsetneq \Omega$ such that

$$\mu_n(\Omega \setminus K_\varepsilon) \leq \varepsilon \quad \forall n \geq N.$$

Let U be a non-empty open subset of $\widehat{\mathbb{C}}$. **Without** appealing to the dominated convergence theorem — but to a more elementary convergence theorem — show that

$$\lim_{n\to\infty} \mu_n(U) = \mu(U).$$

Remark: The reason we regard the dominated convergence theorem as non-elementary — especially in comparison to the theorem you are required to use above — is because for **general** measures on general σ-algebras, its proof is quite non-obvious.