UM 204 : INTRODUCTION TO BASIC ANALYSIS

SPRING 2019

HOMEWORK 10

Instructor: GAUTAM BHARALI **Assigned: MARCH 23, 2019**

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1. Let $r \in \mathbb{R}$ and let p be a positive real number. Consider the function $f : [-1, 1] \longrightarrow \mathbb{R}$ given by:

$$f(x) := \begin{cases} x^r \sin(1/x^p), & \text{if } 0 < x \leq 1, \\ 0, & \text{if } -1 \leq x \leq 0. \end{cases}$$

Find (i) a necessary & sufficient condition on (r, p) for f to be differentiable at 0; (ii) a necessary & sufficient condition on (r, p) for f to be differentiable at 0 **and** such that f' is continuous at 0.
Note. You may assume that the function $\phi_r : x \longmapsto x^r$, $x \in (0, \infty)$ is differentiable on $(0, \infty)$ for any $r \in \mathbb{R}$, and $\phi_r'(x) = rx^{r-1}$ $\forall x \in (0, \infty)$.

2–5. Problems 4–6 and 18 from "Baby" Rudin, Chapter 5.

6. Let I be an open interval and let $f : I \longrightarrow \mathbb{R}$ be an injective differentiable function.

(a) Show that $f(I)$ is an open interval.

(b) Show that f^{-1} is differentiable at each y belonging to the set

$$\mathscr{R}_f := \{y \in f(I) : f'(f^{-1}(y)) \neq 0\}$$

and that

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(f^{-1}(y))} \quad \text{for each } y \in \mathscr{R}_f.$$

Hint. First deduce that f^{-1} is continuous.

7. Find a rational number that approximates $\sqrt{10}$ by

- choosing an appropriate interval $[a, b] \subseteq \mathbb{R}$ and an appropriate function $f : [a, b] \longrightarrow \mathbb{R}$; and
- letting your approximation be the Taylor polynomial $T_f^3(x; x_0)$ — around a suitable $x_0 \in [a, b]$ — evaluated at a convenient integer/rational number, say, $\alpha \in [a, b]$.

Now, with your choices of x_0 and α, give an efficient and **explicit** bound on the error in your approximation: i.e., an efficient upper bound on

$$\left|\sqrt{10} - T_f^3(\alpha; x_0)\right|.$$