UM 204 : INTRODUCTION TO BASIC ANALYSIS

SPRING 2019

HOMEWORK 11

Instructor: GAUTAM BHARALI **Assigned: MARCH 29, 2019**

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1. Define the function $f : \mathbb{R} \longrightarrow \{0, 1\}$ as follows:

$$f(x) := \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Show that $f|_{[a,b]} \notin \mathscr{R}([a, b])$ for any $a < b$.

2–4. Problems 7, 8, and 16 from "Baby" Rudin, Chapter 6.

5. Let $a \leq b \in \mathbb{R}$ and suppose $f : [a, b] \longrightarrow \mathbb{R}$ is Riemann integrable. Let $\alpha, \beta \in \mathbb{R}$ be such that $a \leq \alpha \leq \beta \leq b$. Show that $f|_{[\alpha,\beta]}$ is Riemann integrable on $[\alpha, \beta]$.

6. Let $a < b \in \mathbb{R}$, and let $f, g \in \mathscr{R}([a, b])$. Let p and q be positive real numbers such that $p^{-1} + q^{-1} = 1$. Prove **Hölder's inequality:**

$$\left| \int_a^b fg(x)\, dx \right| \leq \left[\int_a^b |f(x)|^p dx \right]^{1/p} \left[\int_a^b |g(x)|^q dx \right]^{1/q},$$

by completing the outline provided by parts (a)–(c) of Problem 10 in "Baby" Rudin, Chapter 6, taking $\alpha = \mathsf{id}_{[a,b]}$.

7. Prove the following assertion:

> Let $[a, b]$ be a compact interval and let $f : [a, b] \longrightarrow \mathbb{R}$ be a bounded function. Let $\mathcal{D}_f$ denote the set of discontinuities of f. Suppose that, for each $n \in \mathbb{Z}_+$, there exists a pairwise disjoint collection of closed subintervals of $[a, b]$: $\mathscr{C}_n := \{I_1^{(n)}, \dots, I_{M(n)}^{(n)}\}$, such that
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> $$\mathcal{D}_f \subset \bigcup\nolimits_{i=1}^{M(n)} I_j^{(n)} \quad \text{and} \quad \sum_{i=1}^{M(n)} \text{length}\left(I_j^{(n)}\right) \leq \frac{1}{n}$$
>
> for each $n = 1, 2, 3, \dots$ Then $f \in \mathscr{R}([a, b])$.

by using the following hint:

Let $C > 0$ be such that $|f(x)| \leq C$ for each $x \in [a, b]$. Fix $\varepsilon > 0$. Show that there exists a partition on $[a, b]$

$$\mathbb{P} : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

for which we may split the set $\{1, \dots, n\}$ into a disjoint union $G \cup B$ such that

$$\sum_{j \in G} (M_j - m_j)\Delta x_j < \varepsilon/2; \text{ and}$$

$$\sum_{j \in B} \Delta x_j < \frac{\varepsilon}{4C}.$$

Here M_j and m_j have the same meanings as introduced in class in defining the Riemann integral.