UM 204 : INTRODUCTION TO BASIC ANALYSIS

SPRING 2019

HOMEWORK 2

Instructor: GAUTAM BHARALI **Assigned: JANUARY 18, 2019**

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1. Let X and Y be two non-empty sets and let $f, g : X \longrightarrow Y$ be two functions. Why do define $f = g$ as

$$f(x) = g(x) \;\; \forall x \in X\,?$$

Be sure that you give a mathematical reason!

2. Let S be a non-empty set, and let $\sim$ be an equivalence relation on S. Recall that, for any $s \in S$, the *equivalence class of* s — denoted by $[s]$ — is defined as

$$[s] := \{x \in S : x \sim s\}.$$

Prove that $\sim$ partitions S into disjoint equivalence classes.

3. Review the definition, in algebra, of a **field.**

4. Consider a set $A = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$ on which we define two binary operations $+$ and $\cdot$ as follows:

$$\overline{a} + \overline{b} := \overline{(a+b) \bmod 6}, \qquad \overline{a} \cdot \overline{b} := \overline{(a\,b) \bmod 6}. \tag{1}$$

(Given any $a \in \mathbb{N}$, we define "a mod 6" as follows:

a mod 6
:= the unique remainder belonging to the set $\{0, 1, 2, \dots, 5\}$ obtained when dividing a by 6.)

The operations between the unbarred variables a and b in (1) are the usual addition and multiplication on $\mathbb{N}$. Is $(A, +, \cdot)$ a field? Justifiy your answer.

5. Write $\mathbb{F}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$ on which we define the binary operations $+$ and $\cdot$ analogous to those in the previous problem:

$$\overline{a} + \overline{b} := \overline{(a+b) \bmod 3}, \qquad \overline{a} \cdot \overline{b} := \overline{(a\,b) \bmod 3}.$$

Show that $(\mathbb{F}_3, +, \cdot)$ is a field.

Remark. The above holds true with any prime number replacing 3. The non-trivial step, in that case, is to show that multiplicative inverses exist — which follows from a result called *Fermat's Little Theorem.*