UM 204 : INTRODUCTION TO BASIC ANALYSIS

SPRING 2019

HOMEWORK 3

Instructor: GAUTAM BHARALI **Assigned: JANUARY 26, 2019**

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1. The following two problems establish that the operations "+" and "·" defined on $\mathbb{Z}$ extend Peano arithmetic to $\mathbb{Z}$.

(a) Define the function $f : \mathbb{N} \longrightarrow \mathbb{Z}$ by $f(n) := (n -_{\mathbb{Z}} 0)$ for each $n \in \mathbb{N}$. Show that f is injective.

(b) Show that

$$\begin{aligned} f(m+n) &= f(m) + f(n), \\ f(m \cdot n) &= f(m) \cdot f(n), \quad \forall m, n \in \mathbb{N}. \end{aligned}$$

2. Show that $m + (-n) = (m -_{\mathbb{Z}} n)$ for each $m, n \in \mathbb{Z}$.

3. Formulate and prove a pair of statements analogous to those in Problem 1 that establish that the operations "+" and "·" defined on $\mathbb{Q}$ extend the arithmetic on $\mathbb{Z}$.

4. This is an easy problem meant to familiarize you with the "language" and notations used in mathematics. Let S be a non-empty set equipped with a strict order $\prec$. Let

$$\mathsf{diag} := \{(x, x) \in S \times S : x \in S\}.$$

Consider the relation $\preceq := \prec \cup \mathsf{diag}$. Express the statement $x \preceq y$ in terms of x, y and $\prec$, where $x, y \in S$. Is $\preceq$ an order on S?

5. Let $m, n \in \mathbb{N}$. It follows that:

(i) if $m \geq n$, then there is a unique $\mu \in \mathbb{N}$ such that $m = \mu + n$.

(ii) if $n \geq m$, then there is a unique $\mu \in \mathbb{N}$ such that $n = \mu + m$.

Show that $(m -_{\mathbb{Z}} n) = (\mu -_{\mathbb{Z}} 0)$ if (i) holds true and that $(m -_{\mathbb{Z}} n) = (0 -_{\mathbb{Z}} \mu)$ if (ii) holds true.

6. Recall that if α is a positive cut, then we define

$$\alpha^{-1} := \{x \in \mathbb{Q} : \exists r \in \mathbb{Q} \text{ such that } r < 1/x \text{ and } r \notin \alpha\} \cup 0^* \cup \{0\}.$$

(a) Define α^{-1} for a negative cut.

(b) Show that α^{-1} as defined **is** a cut for any $\alpha \neq 0^*$.

7. Let A be a non-empty at most countable set and suppose, for each $\alpha \in A$, we are given a set B_α that is at most countable. *We know that* $S := \bigcup_{\alpha \in A} B_\alpha$ *is at most countable.* Now suppose that A is *countable*, and assume that $B_\alpha \neq B_{\alpha'}$ for $\alpha \neq \alpha'$. Is S countable? If yes, then give a justification, else give a counterexample.

8. Let S be a non-empty set. Show that the power set of S has the same cardinality as the set of all functions from S to the set $\{0, 1\}$.

9. Let S be an uncountable set. Show that:

(a) There exists an injective function from S into $\mathcal{P}(S)$.

(b) S does **not** have the same cardinality as $\mathcal{P}(S)$.

Hint. The conclusions of Problem 8 above might be of help.