UM 204 : INTRODUCTION TO BASIC ANALYSIS

SPRING 2019

HOMEWORK 4

Instructor: GAUTAM BHARALI **Assigned: FEBRUARY 1, 2019**

Note. Problem 1 has been carried over from Assignment 3. Here, given a set S, the *power set* of S — denoted by $\mathcal{P}(S)$ — will refer to the set of all subsets of S.

1. Let S be an uncountable set. Show that:

(a) There exists an injective function from S into $\mathcal{P}(S)$.

(b) S does **not** have the same cardinality as $\mathcal{P}(S)$.

Hint. The conclusions of Problem 8 from Assignment 3 might be of help.

2. Let S be a non-empty subset of $\mathbb{N}$. Show that S contains a unique least element (with respect to the standard order "$\leq$" on $\mathbb{R}$).

3. Prove the density property of $\mathbb{Q}$ in $\mathbb{R}$ using the Archimedean property of $\mathbb{R}$.

4. (Problem 2 from "Baby" Rudin, Chapter 2) A complex number is said to be an *algebraic number* if it is the root of an algebraic equation of the form:

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

for some $n \in \mathbb{Z}_+$, where $a_1, \dots, a_n$ are integers.
Show that the set of all algebraic integers is countable.

5. A *graph* $G := G(V, E)$ is a pair of sets (V, E), where V is a non-empty, at most countable set, and $E \subset T(V)$, where

$$T(V) \ := \ \{\{x, y\} : x, y \in V, \ x \neq y\}.$$

The set V is called the set of *vertices of* G and E is called the set of *edges of* G. Consider the following definitions:

- Given $x \neq y \in V$, a *path joining* x *to* y is a finite collection of edges $\{\{x_j, y_j\} \in E : j = 0, \dots, N\}$ such that $x_0 = x$, $y_{j-1} = x_j, \ j = 1 \dots N$, and $y_N = y$. The *length* of a path is the number of edges contained in it.
- The graph $G(V, E)$ is said to be *connected* if, for each $x \neq y \in V$, there is at least one path joining x to y.
- If $G(V, E)$ is a connected graph, define the function $d : V \times V \longrightarrow [0, \infty)$ by

$$d(x, y) \ = \ \begin{cases} 0, & \text{if } x = y, \\ \min\{\text{length}(P) : P \textit{ is a path joining } x \textit{ to } y\}, & \text{if } x \neq y. \end{cases}$$

Given any connected graph $G = G(V, E)$, is (V, d) a metric space ? If yes, then justify, else give a counterexample.

6. Let (X, d) be a metric space and $\{S_\alpha : \alpha \in A\}$ an arbitrary collection of subsets of X. State whether the correct relation **in general** should be $B \supseteq C$ or $B \subseteq C$ or $B = C$, where

$$B = \bigcup_{\alpha \in A} \overline{S}_\alpha \quad \text{and} \quad C = \overline{\bigcup_{\alpha \in A} S_\alpha}.$$

If $B \neq C$ in general, then provide an example showing that the relevant inclusion could be a strict inclusion.

7. Given a metric space (X, d) and a set $S \subset X$, we say that a point $x \in S$ is an *interior point of* S if there exists an $r > 0$ such that $B(x; r) \subseteq S$. Show that $S^\circ =$ the set of all interior points of S.

8. Show that the density property of $\mathbb{Q}$ in $\mathbb{R}$ (i.e., the property about the usual order "$\leq$" on $\mathbb{R}$ mentioned in Problem 3) is equivalent to saying that $\mathbb{Q}$ is dense in $\mathbb{R}$ endowed with the standard metric.

9. (Problem 24 from "Baby" Rudin, Chapter 2) A metric space is called *separable* if it has a countable dense subset. Now suppose (X, d) is a metric space in which every infinite subset has a limit point. Show that X is separable.