UM 204 : INTRODUCTION TO BASIC ANALYSIS

SPRING 2019

HOMEWORK 5

Instructor: GAUTAM BHARALI **Assigned: FEBRUARY 8, 2019**

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1. Fix a real number $\alpha \in (0,1)$. Let $\mathscr{C}_\alpha$ be a subset of $[0,1]$ having a construction analogous to that of the Cantor middle-thirds set. Namely, let:

$K_0 := [0,1]$,

$K_n :=$ the union of the closed intervals obtained by removing from $K_{n-1}^{(j)}$, $j = 1, \dots, 2^{n-1}$, the open interval of length $\alpha \cdot \mathsf{length}\big(K_{n-1}^{(j)}\big)$ centered at the midpoint of $K_{n-1}^{(j)}$, $n = 1, 2, 3, \dots$,

where $K_{n-1}^{(1)}, \dots, K_{n-1}^{(2^{n-1})}$ are the closed intervals whose union gives K_{n-1}. Emulate what you have studied about the Cantor middle-thirds set to show that:

(*a*) $\mathscr{C}_\alpha$ is non-empty and is a perfect set.

(*b*) the sum of the lengths of the disjoint open intervals removed from $[0,1]$ in the construction of $\mathscr{C}_\alpha$ equals 1.

2. Consider the sequence $\{a_n\}$, where

$$a_n = (n^2+1)^{1/4} - \sqrt{n+1}, \quad n = 1, 2, 3, \dots$$

Does this sequence converge? Give **justifications** for your answer, and if the sequence converges, then deduce its limit.

3. Let $\{a_n\}$ be a complex sequence that converges to A. Then, show that the sequence of arithmetic means

$$\mu_n := \frac{a_1 + \cdots + a_n}{n}, \quad n = 1, 2, 3, \dots ,$$

is also convergent.
Tip. It would help to guess what $\{\mu_n\}$ converges to!

4. Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X. Show that $\{x_n\}$ converges to a point $x_0 \in X$ if and only if every subsequence $\{x_{n_k}\}_{k \in \mathbb{Z}_+}$ converges to x_0.

5–7. Problems 20–22 from "Baby" Rudin, Chapter 3.

8. Fix a real number $a > 0$. Recall that if $p \in \mathbb{Q}$ and if $p = m/n$ for some $m \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, then

$$a^p := (a^m)^{1/n}. \tag{1}$$

Given this, if now p is an arbitrary real number, then we define

$$a^p := \sup\{a^q : q \in \mathbb{Q} \text{ and } q \le p\}. \tag{2}$$

Assuming, for the moment, that the definition (1) does not depend on the choice of the representative m/n of p, show that:

(a) for $p \in \mathbb{Q}$ the two definitions (1) and (2) of a^p agree.

(b) for any $x, y \in \mathbb{R}$, $a^x a^y = a^{x+y}$.