UM 204 : INTRODUCTION TO BASIC ANALYSIS

SPRING 2019

HOMEWORK 6

Instructor: GAUTAM BHARALI **Assigned: FEBRUARY 15, 2019**

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1. In each of the following cases, determine using **only** the basic definition whether or not S is a compact subset of X (i.e., do **not** appeal to any theorems on compactness):

(a) $(X, d) =\mathbb{R}$ equipped with the standard metric; $S = \{1/n = n : 1, 2, 3, \dots\} \cup \{0\}$.

(b) $(X, d) =$ any set containing at least two points, equipped with the 0–1 metric; $S \subset X$ (your answer will depend on the nature of S; please give a **complete** discussion).

(c) The interval $[0, 1)$ in the real line.

2. (Problem 24 from "Baby" Rudin, Chapter 2) A metric space is called *separable* if it has a countable dense subset. Now suppose (X, d) is a metric space in which every infinite subset has a limit point. Show that X is separable.

3. Show that if S is a non-empty subset of $\mathbb{R}$ that is bounded above, then $\sup S$ ~~is a limit point of S~~ belongs to $\overline{S}$.

4. Show that any bounded sequence in $\mathbb{R}$ has a convergent subsequence.

5. Fix a real number $a > 0$ and a number $p \in \mathbb{R} \setminus \mathbb{Q}$. Let $\{q_n\}$ be any sequence of rational numbers such that $q_n \to p$.

(a) Show that $\{a^{q_n}\}$ is a convergent sequence.

Hint. It suffices to show that $\{a^{q_n}\}$ is Cauchy. Also, you may assume **without proof** that $\lim_{n\to\infty} a^{1/n} = 1$.

(b) Show that the limit of $\{a^{q_n}\}$ is a^p as defined in the statement of Problem 8 from Assignment 5.