UM 204 : INTRODUCTION TO BASIC ANALYSIS

SPRING 2019

HOMEWORK 7

Instructor: GAUTAM BHARALI **Assigned: MARCH 2, 2019**

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1. Let $\{a_n\}$ be a sequence of real numbers. Define

$$A_k := \inf\{a_k, a_{k+1}, a_{k+2}, \dots\},$$
$$B_k := \sup\{a_k, a_{k+1}, a_{k+2}, \dots\}.$$

Show that

$$\liminf_{n\to\infty} a_n = \lim_{k\to\infty} A_k, \quad \text{and} \quad \limsup_{n\to\infty} a_n = \lim_{k\to\infty} B_k.$$

2. Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$ and let $\|\boldsymbol{\cdot}\|$ be a norm on V. If we define $d(v, w) := \|v - w\|$, then show that d is a metric on V.

3. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with terms in $\mathbb{R}$. Let $\{b_n\} \subset \mathbb{R}$ be a monotone bounded sequence. Show that $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

4. Let V be a vector space over the field $\mathbb{F}$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, equipped with a norm $\|\boldsymbol{\cdot}\|$. Let

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} b_n$$

be convergent series with terms in V whose sums are A and B respectively. Let $c \in \mathbb{F}$. Show that:

- the series $\sum_{n=1}^{\infty}(a_n + b_n)$ is convergent and its sum is $(A + B)$.
- the series $\sum_{n=1}^{\infty} ca_n$ is convergent and its sum is cA.

5. Complete the following outline for a proof that the interval $[0, 1)$ is uncountable. Given a number $x \in [0, 1)$, let $\mathcal{I}_0(x) := [0, 1)$ and define the intervals

$$\mathcal{I}_{n+1}(x) := \begin{cases} \big[\inf \mathcal{I}_n(x), \mu_n(x)\big), & \text{if } x < \mu_n(x), \\ \big[\mu_n(x), \sup \mathcal{I}_n(x)\big), & \text{if } x \geq \mu_n(x), \end{cases}$$

for $n = 0, 1, 2, \dots$, where $\mu_n(x) := \big(\inf \mathcal{I}_n(x) + \sup \mathcal{I}_n(x)\big)/2$: i.e., the midpoint of $\mathcal{I}_n(x)$. Let $\mathfrak{S}$ denote the set of all sequences in $\{0, 1\}$. We now define a function $F : [0, 1) \to \mathfrak{S}$ as follows: write $F(x) = \{s_n(x)\}$ where

$$s_n(x) := \begin{cases} 0 & \text{if } x < \mu_{n-1}(x), \\ 1, & \text{if } x \geq \mu_{n-1}(x), \end{cases}$$

for $n = 1, 2, 3, \dots$.

(*a*) Show that the series

$$\sum_{n=1}^{\infty} \frac{s_n(x)}{2^n}$$

converges, and that its sum is x. (**Remark.** This problem shows that the "binary representation" of x — i.e., the expression "$0.s_1(x)\, s_2(x)\, s_3(x) \dots$", which is analogous to the common decimal expressions for real numbers — exists.)

(b) Show that F is **not** surjective (use the conclusion of (a) above).

(c) Show that $\mathfrak{S} \setminus \mathsf{range}(F)$ is countable.

(d) Use the conclusions of (a)–(c) to show that $[0,1)$ is uncountable.

6. Consider the series

$$\sum\nolimits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n},$$

and let S_n denote its n-th partial sums, $n = 1, 2, 3, \ldots$

(a) Show that the sequences $\{S1, S_3, S_5, \ldots\}$ and $\{S_2, S_4, S_6, \ldots\}$ are monotone and bounded.

(b) Use the conclusions of (b) to deduce that the above series converges.