UM 204 : INTRODUCTION TO BASIC ANALYSIS

SPRING 2019

HOMEWORK 8

Instructor: GAUTAM BHARALI **Assigned: MARCH 8, 2019**

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1. Any rational number x can be written uniquely as $x = m/n$, where $m \in \mathbb{Z}$, $n \in \mathbb{Z}_+$, and such that there is no $d \in \mathbb{N} \setminus \{0, 1\}$ dividing both m and n — with the understanding that we take $n = 1$ when $x = 0$. (You may use this fact **without proof.** You have learnt in UM203 what "d divides m (or n)" means.) Define $f : \mathbb{R} \to \mathbb{Q}$ as follows:

$$f(x) := \begin{cases} 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ 1/n, & \text{if } x \in \mathbb{Q}, \end{cases}$$

where n is uniquely associated to $x \in \mathbb{Q}$ as explained above. Show that f is continuous at each irrational point and discontinuous at each rational point.

2. Let (X, d_X) and (Y, d_Y) be metric spaces, $S \subseteq X$, and let $f : S \longrightarrow Y$ be a function. Show that f is continuous at each isolated point of S.

Note. If $a \in S$ is an isolated point, then the class of sequences in $S \setminus \{a\}$ that converge to a is vacuous. So, intuitively, one expects the above owing to the truth of vacuous implications. However, we **cannot** appeal to the sequential definition for the limit of f at a since that definition is valid only at limit points of S! Thus, a **formal** proof would require a different approach.

3. Let (X, d) be a metric space and suppose $f : X \longrightarrow \mathbb{R}$ is a function that maps any Cauchy sequence in X to a Cauchy sequence. Show that f is continuous. Can you state a more general form of this result?

4. Let $n \geq 2$, $n \in \mathbb{Z}_+$. Prove from first principles (i.e., without using any results on sums/products of continuous functions), that $f(x) = x^n$, $x \in \mathbb{R}$, is continuous on $\mathbb{R}$.

5. Let $q \in \mathbb{Q}$ be a fixed rational number. Show that $f(x) = x^q$, $x \in [0, +\infty)$, is continuous on $[0, +\infty)$.

Note. It might not be pleasant to prove the above **entirely** from first principles!

6. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f, g : X \longrightarrow Y$ be two functions. Let $S \subseteq X$ be a dense subset. Suppose $f(x) = g(x)$ for each $x \in S$. Show that $f \equiv g$.