UM 204 : INTRODUCTION TO BASIC ANALYSIS

SPRING 2019

HOMEWORK 9

Instructor: GAUTAM BHARALI **Assigned: MARCH 16, 2019**

The following is the **corrected** version of a problem assigned in Homework 6.

1. Show that if S is a non-empty subset of $\mathbb{R}$ that is bounded above, then $\sup S$ belongs to $\overline{S}$.

2. Let X be a metric space and let $S \subseteq X$. Show that S is connected if and only if there do **not** exist two non-empty subsets $A, B \subseteq S$ such that $S = A \cup B$ and A and B are separated (in X).

3. Let S_1 and S_2 be non-empty sets and let $f : S_1 \longrightarrow S_2$. Let $\mathscr{A}$ be a non-empty subset of $\mathcal{P}(S_1)$ and let $\mathscr{B}$ be a non-empty subset of $\mathcal{P}(S_2)$. Prove the following:

$$\begin{aligned} f\big(\cup_{A\in\mathscr{A}} A\big) &= \cup_{A\in\mathscr{A}} f(A), \\ f\big(\cap_{A\in\mathscr{A}} A\big) &\subseteq \cap_{A\in\mathscr{A}} f(A), \\ f^{-1}\big(\cup_{B\in\mathscr{B}} B\big) &= \cup_{B\in\mathscr{B}} f^{-1}(B), \\ f^{-1}\big(\cap_{B\in\mathscr{B}} B\big) &= \cap_{B\in\mathscr{B}} f^{-1}(B). \end{aligned}$$

4. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \longrightarrow \mathbb{R}$ be a continuous function. Show that f is invertible if and only if it is either strictly increasing or strictly decreasing.

5. Let X and Y be metric spaces and let $f : X \longrightarrow Y$ be a continuous invertible function. If X is compact, then show that f^{-1} is continuous on $\mathsf{range}(f)$.

6. Give a proof of Brouwer's Fixed Point Theorem in the $n = 1$ case.

7–8. Problems 20 and 22 from "Baby" Rudin, Chapter 4.

9. Consider the result:

Theorem. *Let X and Y be metric spaces, and let $S \subsetneq X$ be dense subset. Let $f : S \longrightarrow Y$ be a uniformly continuous function. Suppose Y is complete. Then, there exists a continuous function $\widetilde{f} : X \longrightarrow Y$ such that $\widetilde{f}\big|_S = f$.*

that was *partially* proved in class. Consider the function $\widetilde{f}$ constructed in that proof — which must be shown to have the properties stated above. Fix $x \in (X \setminus S)$, and let $\{x_n\}$ be a sequence in $X \setminus \{x\}$ that converges to x. Complete the following outline to prove that $\widetilde{f}$ is continuous:

(a) Explain why it suffices to only consider sequences $\{x_n\}$ such that

$$\mathsf{range}(\{x_n\}) \bigcap (X \setminus S) \text{ is an infinite set.} \tag{1}$$

(b) Consider a sequence $\{x_n\}$ with the property (1). Construct an auxiliary sequence $\{y_n\} \subset S$ such that for each n for which $x_n \notin S$, y_n is "sufficiently close" to x_n — in an appropriate sense — and converges to x in such a way that you can use its behaviour, plus uniform continuity, to infer that $\{\widetilde{f}(x_n)\}$ is convergent.

(*c*) Deduce that $\{\widetilde{f}(x_n)\}$ converges to $\widetilde{f}(x)$.

(*d*) Now, complete the argument showing that $\widetilde{f}$ is continuous.

10. Give an example showing that, with X, Y and $S \subseteq X$ exactly as in the statement of the theorem in Problem 8 and $f : S \longrightarrow Y$ a continuous function, the theorem is **false** if f is not assumed to be **uniformly** continuous.