UM 204 : INTRODUCTION TO BASIC ANALYSIS

SPRING 2019

QUIZ 1 **FEBRUARY 1, 2019**

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1. Let $m, n \in \mathbb{N}$. It follows that:

(i) if $m \geq n$, then there is a unique $\mu \in \mathbb{N}$ such that $m = \mu + n$.

(ii) if $n \geq m$, then there is a unique $\mu \in \mathbb{N}$ such that $n = \mu + m$.

Show that $(m -_{\mathbb{Z}} n) = (\mu -_{\mathbb{Z}} 0)$ if (i) holds true and that $(m -_{\mathbb{Z}} n) = (0 -_{\mathbb{Z}} \mu)$ if (ii) holds true.
Note. We have not proved in class the uniqueness of the μ appearing in (i) and (ii). However, for this quiz, you are **not required** to establish uniqueness of μ.

Solution. This problem just requires understanding the definition of the equivalence relation $\sim_{\mathbb{Z}}$ on $\mathbb{N} \times \mathbb{N}$. We need to show that

$$(i) \Rightarrow (m -_{\mathbb{Z}} n) \text{ and } (\mu -_{\mathbb{Z}} 0) \text{ denote the same equivalence class under } \sim_{\mathbb{Z}}, \text{ and} \tag{1}$$
$$(ii) \Rightarrow (m -_{\mathbb{Z}} n) \text{ and } (0 -_{\mathbb{Z}} \mu) \text{ denote the same equivalence class under } \sim_{\mathbb{Z}}. \tag{2}$$

To this end, we assume (i). Then:

$$\begin{aligned} m = \mu + n \;&\Rightarrow\; m + 0 = n + \mu \\ &\Rightarrow\; (m, n) \sim_{\mathbb{Z}} (\mu, 0) \qquad [\text{by definition of } \sim_{\mathbb{Z}}]. \end{aligned}$$

This establishes (1)

Next, we assume (ii). Then:

$$\begin{aligned} n = \mu + m \;&\Rightarrow\; 0 + n = \mu + m \\ &\Rightarrow\; (0, \mu) \sim_{\mathbb{Z}} (m, n) \qquad [\text{by definition of } \sim_{\mathbb{Z}}]. \end{aligned}$$

This establishes (2). □