UM 204: INTRODUCTION TO BASIC ANALYSIS
SPRING 2019

QUIZ 2 **FEBRUARY 8, 2019**

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1. Let (X, d) be a metric space and $\{S_\alpha : \alpha \in A\}$ an arbitrary collection of subsets of X. State whether the correct relation **in general** should be $B \supseteq C$ or $B \subseteq C$ or $B = C$, where

$$B = \bigcup_{\alpha \in A} \overline{S}_\alpha \quad \text{and} \quad C = \overline{\bigcup_{\alpha \in A} S_\alpha}.$$

If $B \neq C$ in general, then provide an example showing that the relevant inclusion could be a strict inclusion.

Solution. Let $x \in B$. Then, there exists an α^* (which depends on x) such that $x \in \overline{S}_{\alpha^*}$. Then, it follows from the theorem characterizing the closure of any set in a metric space that either $x \in S_{\alpha^*}$ or is a limit point of S_{α^*}. In the first case

$$x \in \bigcup_{\alpha \in A} S_\alpha \subseteq C.$$

In the second case, given any $r > 0$

$$B(x; r)\setminus\{x\} \cap S_{\alpha^*} \neq \varnothing$$
$$\Rightarrow B(x; r)\setminus\{x\} \cap \left(\bigcup_{\alpha \in A} S_\alpha \right) \neq \varnothing$$

which implies that $x \in C$. In either case, thus, $x \in C$. Since $x \in B$ was arbitrary, we have established that $B \subseteq C$.

The opposite inclusion is **false**. Consider, for example, (X, d) to be $\mathbb{R}$ with the standard metric, and consider the collection

$$\big\{[1/n, 1 - 1/n] : n \in \mathbb{Z}_+\big\}.$$

I.e., $S_n = [1/n, 1 - 1/n] = \overline{S}_n$. We make the following

Claim. $B = (0, 1)$

If $x \in B$ then there exists some $n^* \in \mathbb{Z}_+$ such that $1/n^* \leq x \leq 1 - 1/n^*$, whence $0 < x < 1$. If $x \in (0, 1)$, then we can find an integer $n_1 \in \mathbb{Z}_+$ such that $1/n_1 < x$ (by an argument given in class using the Archimedean property of $\mathbb{R}$). Also, we can find an integer $n_2 \in \mathbb{Z}_+$ such that $x < 1 - 1/n_2$ (by the above-mentioned argument applied to $1 - x$). Write $n_0 := \max(n_1, n_2)$. Then $x \in [1/n_0, 1 - 1/n_0]$, and hence $x \in B$.

Since $B = \cup_{n \in \mathbb{Z}_+} S_n$ in the present example, and B is not closed, $B \subsetneq C$. □