UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2019

QUIZ 3 **MARCH 8, 2019**

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1. Let $\{a_n\}$ be a **bounded** sequence of real numbers. Define

$$B_k := \sup\{a_k, a_{k+1}, a_{k+2}, \dots\}.$$

Show that

$$\limsup_{n\to\infty} a_n = \lim_{k\to\infty} B_k.$$

Remark. The hypothesis of $\{a_n\}$ being bounded is **unnecessary.** It has only been provided to reduce the cases one would need to comment upon in general.

Solution. Write $L := \limsup_{n\to\infty} a_n$. Since, by definition

$$\limsup_{n\to\infty} a_n \in \{-\infty, +\infty\} \iff \mathsf{range}[\{a_n\}] \text{ is unbounded,}$$

L must, by hypothesis, be finite. Thus, given $\varepsilon > 0$, there exists a positive integer N_1 such that

$$\begin{aligned} a_n &< L + (\varepsilon/2) \quad \forall n \geq N_1 \\ \Rightarrow B_n &< L + \varepsilon \quad \forall n \geq N_1. \end{aligned} \tag{1}$$

We know that L is itself a subsequential limit: i.e., there is a sequence $\{a_{n_j}\}_{j\in\mathbb{Z}_+}$ such that $\lim_{j\to\infty} a_{n_j} = L$. Thus, there is a positive integer J such that

$$L - \varepsilon < a_{n_j} < L + \varepsilon \quad \forall j \geq J.$$

Write $N_2 := n_J$. The above inequality implies that

$$B_n > L_\varepsilon \quad \forall n \geq n_j = N_2. \tag{2}$$

Write $N := \max(N_1, N_2)$. From (1) and (2), we get

$$|B_n - L| < \varepsilon \quad \forall n \geq L.$$

Since $\varepsilon > 0$ was arbitrarily chosen, we conclude that $\lim_{n\to\infty} B_n = L$. □