UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2019

QUIZ 4 **MARCH 15, 2019**

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1. Let $n \geq 2$, $n \in \mathbb{Z}_+$. Show that $f(x) = x^{1/n}$, $x \in [0, +\infty)$, is continuous on $[0, +\infty)$.

Remark. This forms one of the **parts** to solving Problem 5 from the most recent homework assignment.

Solution. We must first establish the following:

Claim: *f is strictly increasing.*

Define $\varphi_k : [0, +\infty) \to \mathbb{R}$ by $\varphi_k(x) := x^k$. Assume that φ_k is strictly increasing for some $k \in \mathbb{Z}_+$. Then for any $x, y \in [0, +\infty)$:

$$\begin{aligned} x < y \Rightarrow\ & x^k < y^k && \text{[by assumption]} \\ \Rightarrow\ & x^k y < y^k y && \text{[since } y > 0\text{]} \\ \Rightarrow\ & x^k x \leq x^k y < y^k y. \end{aligned}$$

The last two inequalities are immediate consequences of $\mathbb{R}$ being an ordered field. The above calculation shows that if φ_k is strictly increasing, then so is φ_{k+1}. Since $\varphi_1 = \mathsf{id}_{[0,+\infty)}$ is strictly increasing, it follows by mathematical induction that φ_n is strictly increasing. Now, suppose there exist $x, y \in [0, +\infty)$ such that

$$x < y \quad \text{and} \quad f(x) \geq f(y).$$

Then, as φ_n is strictly increasing, we get $x = \varphi_n\big(f(x)\big) \geq \varphi_n\big(f(y)\big) = y$, which is a contradiction. Thus f is strictly increasing.

Fix a point $a \in [0, +\infty)$ and let $\varepsilon > 0$.

If $a = 0$, then take $\delta = \varepsilon^n$. By the above Claim:

$$0 \leq (x - a) = |x| < \delta \Rightarrow 0 \leq f(x) = |f(x) - f(a)| < \varepsilon. \tag{1}$$

If $a > 0$, write $\varepsilon^* := \min\{a^{1/n},\ \varepsilon\}$ and take

$$\delta := \min\{(a^{1/n} + \varepsilon^*)^n - a,\ a - (a^{1/n} - \varepsilon^*)^n\}.$$

Then, by the above Claim:

$$\begin{aligned} |x - a| < \delta \Rightarrow\ & (a^{1/n} - \varepsilon^*)^n < x < (a^{1/n} + \varepsilon^*)^n \\ \Rightarrow\ & -\varepsilon^* < f(x) - f(a) < \varepsilon^* \\ \Rightarrow\ & |f(x) - f(a)| < \varepsilon^* \leq \varepsilon. \end{aligned} \tag{2}$$

Since $\varepsilon > 0$ was chosen arbitrarily, (1) and (2) imply that f is continuous at a. Since a was chosen arbitrarily, it follows that f is continuous. □