UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2019

QUIZ 5 **MARCH 22, 2019**

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1. Let X and Y be metric spaces and let $f : X \longrightarrow Y$ be a continuous invertible (i.e., injective) function. If X is compact, then show that $f^{-1} : \mathsf{range}(f) \longrightarrow X$ is continuous.

Solution. Since $(f^{-1})^{-1}(S) = S$ for any $S \subseteq X$, it suffices to show the following:

($*$) For any open set $U \subset X$, $f(U)$ is open relative to $\mathsf{range}(f)$.

Therefore, consider an arbitrary open set $U \subseteq X$. Since $X \setminus U$ is closed and X is compact, so is $X \setminus U$. Thus, by the continuity of f,

$$
\begin{aligned}
f(X \setminus U) \; &= \; f(X) \setminus f(U) \text{ is compact} &(1)\\
&\Rightarrow \; f(X) \setminus f(U) \text{ is a closed subset of } Y. &(2)
\end{aligned}
$$

The equality in (1) is a consequence of f being injective.

Let us write $E := f(X) \setminus f(U)$. Observe that by definition:

$$
\begin{aligned}
\mathsf{range}(f) \cap (Y \setminus E) \; &= \; \mathsf{range}(f) \cap \{y \in Y : y \in f(U) \text{ or } y \notin f(X)\}\\
&= \; f(U).
\end{aligned}
$$

In view of (2) and the last identity, we see that $f(U)$ is the intersection of some open subset of Y with $\mathsf{range}(f)$. Hence, $f(U)$ is open relative to $\mathsf{range}(f)$. This establishes the statement ($*$), and hence the result. $\square$