UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2019

QUIZ 6 **APRIL 5, 2019**

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1. Define the function $f : \mathbb{R} \longrightarrow \{0, 1\}$ as follows:

$$f(x) := \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Show that $f|_{[a,b]} \notin \mathscr{R}([a, b])$ for any $a < b$.

Solution. We first establish two claims:

Claim 1: *Given any $\alpha < \beta \in \mathbb{R}$, $[\alpha, \beta] \cap \mathbb{Q} \neq \varnothing$.*

Suppose this is false. Then there exist real numbers $\alpha < \beta$ such that $[\alpha, \beta] \cap \mathbb{Q} = \varnothing$. Then $\mathbb{Q} \subseteq (-\infty, \alpha) \cup (\beta, +\infty)$. The density of $\mathbb{Q}$ in $\mathbb{R}$ implies that $\mathbb{R} \subseteq (-\infty, \alpha] \cup [\beta, +\infty)$, which is impossible. Hence the claim.

Claim 2: *Given any $\alpha < \beta \in \mathbb{R}$, $[\alpha, \beta] \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \varnothing$.*

Clearly, $[\alpha, \beta]$ is in one-to-one correspondence with $[0, 1]$. We have shown that $[0, 1]$ is uncountable, whence $[\alpha, \beta]$ is uncountable. Now

$$[\alpha, \beta] \cap (\mathbb{R} \setminus \mathbb{Q}) = [\alpha, \beta] \setminus \mathbb{Q}.$$

Since $\mathbb{Q}$ is countable, $[\alpha, \beta] \setminus \mathbb{Q}$ is uncountable. Thus, $[\alpha, \beta] \cap (\mathbb{R} \setminus \mathbb{Q})$ is uncountable, and hence non-empty.

Now fix $a < b$, and consider a partition

$$\mathbb{P} : \quad a = x_0 < x_1 < x_2 < \cdots < x_N = b.$$

By Claim 1, $[x_{j-1}, x_j] \cap \mathbb{Q} \neq \varnothing$ for $j = 1, \dots N$. Hence, by definition of f:

$$\sup \big\{ f(x) : x \in [x_{j-1}, x_j] \big\} = 1 \quad \forall j = 1, \dots N.$$

Similarly, by Claim 2,

$$\inf \big\{ f(x) : x \in [x_{j-1}, x_j] \big\} = 0 \quad \forall j = 1, \dots N.$$

From the last two equations, we get

$$L(\mathbb{P}, f) = 0 \quad \text{and} \quad U(\mathbb{P}, f) = 1.$$

Since the partition $\mathbb{P}$ was arbitrarily chosen, it follows that the upper and lower integrals of f on $[a, b]$ do not agree. Hence, $f|_{[a,b]} \notin \mathscr{R}([a, b])$. □