UMA 101 : ANALYSIS & LINEAR ALGEBRA – I
AUTUMN 2023

HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 10 PROBLEMS

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1. Consider the following:

Theorem. *Let $I \subseteq \mathbb{R}$ be a non-empty open interval, $f : I \to \mathbb{R}$ a one-one function, and let $a \in I$. Suppose f is continuous and that f is differentiable at a. Moreover, assume that $f'(a) \neq 0$. Then:*

(i) *$f(I)$ is an open interval.*

(ii) *f^{-1} is differentiable at $f(a)$ and*

$$(f^{-1})'\big(f(a)\big) = 1/f'(a).$$

Keeping in mind the discussion in class, give a proof — using the sequential definition of limits of functions — of part *(ii)*.

2. Let $I \subseteq \mathbb{R}$ be a non-empty open interval and let $f : I \to \mathbb{R}$ be a one-one function. Assume that f is bounded and continuous on I. Show that $f(I)$ is an open interval.

Remark. The above is a proof, for a special case, of part (i) of Problem 1. The general result is somewhat annoying to prove using **only** the techniques presented in this course.

Sketch of solution: Since f is a bounded function, by definition

$$f(I) := \{f(x) \in \mathbb{R} : x \in I\}$$

is bounded above and bounded below. Thus, by the least upper bound property of $\mathbb{R}$, $\beta := \sup f(I)$ exists. We have seen that $\mathbb{R}$ also has the greatest lower bound property. Thus, $\alpha := \inf f(I)$ exists. We will show that $f(I) = (\alpha, \beta)$.

At this stage we will need a result presented in class which states that f, under the given hypothesis, is either strictly increasing or strictly decreasing. WLOG, we can assume that f is strictly increasing.

Let $y \in (\alpha, \beta)$. Since $\alpha < y < \beta$, y is neither an upper bound nor a lower bound of $f(I)$, there exist $p_1 < y$ such that $p_1 \in f(I)$ and $p_2 > y$ such that $p_2 \in f(I)$. But $p_1, p_2 \in f(I)$ means that p_1 and p_2 are two distinct values of f. As $p_1 < y < p_2$, by the intermediate-value theorem (recall that f is continuous),

$$\exists c \in (f^{-1}(p_1), f^{-1}(p_2)) \text{ such that } y = f(c).$$

Thus $y \in f(I)$. But since y was chosen arbitrarily, we conclude that

$$(\alpha, \beta) \subseteq f(I). \tag{1}$$

Now let $y \in f(I)$. Then, since α (respectively, β) is a lower (respectively, upper) bound of $f(I)$, $\alpha \leq y \leq \beta$. We first show that $y \neq \beta$. To see this, assume $y = \beta$. Then, as $y \in f(I), \exists p \in I$ such that $f(p) = y = \beta$. Since I is an **open** interval, there exists $\epsilon > 0$ such that $(p-\epsilon, p+\epsilon) \subseteq I$. Then $(p+\epsilon/2) \in I$ and we have

$$f(I) \ni f(p+\epsilon/2) > f(p) = \beta \qquad [\text{as } f \text{ is strictly increasing}]$$

But this contradicts the fact that $\beta = \sup f(I)$. Hence $y < \beta$.

Give a similar argument showing that $y > \alpha$. The last two assertions imply that $y \in (\alpha, \beta)$. But since y was chosen arbitrarily from $f(I)$,

$$f(I) \subseteq (\alpha, \beta). \tag{2}$$

From (1) and (2), the result follows.

3. This problem recapitulates the discussion in class — leading to the computation of the derivative of $\sin^{-1}$ — for the function $\cos^{-1}$.

a) Write down **all** the closed intervals $I \subsetneq \mathbb{R}$ of length π such that $\cos|_I$ is invertible.

b) Define the function $\cos^{-1}$ as follows:

$$\cos^{-1} := \text{the inverse of the function } \cos|_{[0,\pi]}.$$

(This function is also denoted by arccos.) Show that $\cos^{-1}$ is differentiable on $(-1, 1)$ and that

$$(\cos^{-1})'(x) = -\frac{1}{\sqrt{1-x^2}} \quad \forall x \in (-1, 1).$$

4. Fix $n \in \mathbb{N} - \{0, 1\}$ and define $g_n(y) := y^{1/n}$ for each $y \in [0, \infty)$. Using the fact that $g_n = \left(f_n|_{[0,\infty)}\right)^{-1}$ — where $f_n(x) = x^n$ for each $x \in \mathbb{R}$ — show that g_n is differentiable on $(0, \infty)$ and compute $(g_n)'$.

*Sketch of solution:*Fix $y \in (0, \infty)$. Since $g_n = (f_n|_{[0,\infty)})^{-1}$, and since for any $x \in (0, \infty)$, $f_n'(x) = nx^{n-1} > 0$,

$$g_n'(f_n(x)) = \frac{1}{nx^{n-1}} \quad \forall x \in (0, \infty). \tag{3}$$

Since any $y \in (0, \infty)$ is of the form $f_n(x)$, $x \in (0, \infty)$, because clearly

$$y = f_n(y^{1/n}),$$

by (3) we have:

$$g_n'(y) = \frac{1}{n(y^{1/n})^{n-1}}) = \frac{1}{ny^{(n-1)/n}} = \frac{y^{(1/n)-1}}{n} \quad \forall y \in (0, \infty).$$

5. Let $a_1, a_2, \ldots, a_n$ be n **distinct** real numbers. Let

$$f(x) = \sum_{j=1}^{n} (x - a_j)^2, \quad x \in \mathbb{R}.$$

Show that the least value of f is obtained at the arithmetic mean of $a_1, \ldots, a_n$.

6. Let $a < b$ be real numbers and let $s, t : [a, b] \to \mathbb{R}$ be two simple functions. Go through the following outline to show that $s + t$ is also a simple function.

(*a*) Let

$$\mathcal{P}_1 : a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$
$$\mathcal{P}_2 : a = y_0 < y_1 < y_2 < \cdots < y_m = b$$

be partitions that determine s and t, respectively. Consider the partition $\mathcal{P}_1 \cup \mathcal{P}_2$ (which is called the *common refinement of* $\mathcal{P}_1$ *and* $\mathcal{P}_1$), and denote it as

$$\mathcal{P}_1 \cup \mathcal{P}_2 : a = z_0 < z_1 < z_2 < \cdots < z_N = b.$$

Fix an index l such that $1 \leq l \leq N$. You may assume **without proof** (the proof is annoying, involving the consideration of several cases) that there exist **unique** integers $i(l),\ j(l)$, $1 \leq i(l) \leq n$ and $1 \leq j(l) \leq m$ such that

$$(z_{l-1}, z_l) \;=\; \big(x_{i(l)-1}, x_{i(l)}\big) \cap \big(y_{j(l)-1}, y_{j(l)}\big).$$

(*b*) Let $\sigma_1, \ldots, \sigma_n$ be the values taken by s on the open sub-intervals given by $\mathcal{P}_1$ and $\tau_1, \ldots \tau_m$ be the values taken by t on the open sub-intervals given by $\mathcal{P}_2$. Use Part (*a*) and the latter information to show that $s + t$ is also a step function.

Sketch of solution: The solution to this problem is much simpler than the material building up to it would suggest! We consider the partition $\mathcal{P}_1 \bigcup \mathcal{P}_2$ and let $z_l, l = 0, \ldots, N$, be as given by part (a). As

$$(z_{l-1}, z_l) \;=\; \big(x_{i(l)-1}, x_{i(l)}\big) \cap \big(y_{j(l)-1}, y_{j(l)}\big),$$

we have from the data given:

$$s(x) = \sigma_{i(l)} \;\; \forall x \in (z_{l-1}, z_l),$$
$$t(x) = \tau_{j(l)} \;\; \forall x \in (z_{l-1}, z_l).$$

Thus:

$$(s + t)(x) = \sigma_{i(l)} + \tau_{j(l)} \;\; \forall x \in (z_{l-1}, z_l) \text{ and } l = 1, \ldots, N,$$

which exactly fits the definition of a simple function.

7. Solve parts (*c*)–(*f*) of Problem 1 of Section 1.15 of Apostol.

Sketch of solution: The key to solving these problems is to establish that each of the functions is a step function. Then $\int_a^b f(x)\,dx$ in each case is given by the formula for the integral.

We will present a solution of just one of the problems: Part 1(*c*). Note:

$$[x] = \begin{cases} -1, & \text{if } -1 \leq x \leq 0, \\ 0, & \text{if } 0 \leq x \leq 1, \\ 1, & \text{if } 1 \leq x \leq 2, \\ 2, & \text{if } 2 \leq x \leq 3, \end{cases} \qquad [x + 1/2] = \begin{cases} -1, & \text{if } -1 \leq x \leq -\frac{1}{2}, \\ 0, & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2}, \\ 1, & \text{if } \frac{1}{2} \leq x \leq \frac{3}{2}, \\ 2, & \text{if } \frac{3}{2} \leq x \leq \frac{5}{2}, \\ 3, & \text{if } \frac{5}{2} \leq x \leq 3. \end{cases}$$

This suggests the following defining partition for $f(x) := [x] + [x + \frac{1}{2}]$, $-1 \leq x \leq 2$:

$$\mathcal{P} : -1 < -1/2 < 0 < \cdots < 5/2 < 3 \;\equiv\; x_0 < x_1 < \ldots x_8,$$

for the values of $f(x)$ on the j-th open subinterval are $-2, -1, 0, 1, \ldots, 5$, respectively, $j = 1, \ldots, 8$. By definition

$$\begin{aligned} \int_{-1}^{3} f(x)\,dx &= \sum_{j=1}^{8} \left(f|_{(x_{j-1}, x_j)} \right)(x) \Delta x_j \\ &= 6. \end{aligned}$$

8. Compute the integrals $\int_0^3 [x^2]\,dx$ and $\int_0^9 [\sqrt{x}\,]\,dx$. (Exactly as encountered in the previous problem, given $x \in \mathbb{R}$, $[x]$ denotes the greatest integer $\leq x$.)

Sketch of solution: The same insight as for Problem 7 applies to this problem as well. The integrals are elementary. The detailed solution above to part 1(c) of the problem set 1.15 in Apostol gives you a template for presenting your solution systematically.