UMA 101 : ANALYSIS & LINEAR ALGEBRA – I
AUTUMN 2023

HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 4 PROBLEMS

Instructor: GAUTAM BHARALI **Assigned: AUGUST 29, 2023**

---

1. Let $\{a_n\}$ be a sequence in $\mathbb{R}$. We say "$\{a_n\}$ is bounded" if the **set** $\{a_n : n = 1, 2, 3, \dots\}$ is bounded above and bounded below. Prove that if $\{a_n\}$ converges, then it is bounded.

Tip. If $\{c_1, c_2, \dots, c_N\} \subset \mathbb{R}$ is a finite set, then you may freely use $\max\{c_1, c_2, \dots, c_N\}$ — if required — without spelling out its definition (which states exactly what you understood by it in school) or proving that $\max\{c_1, c_2, \dots, c_N\}$ exists.

Sketch of solution: As $\{a_n\}$ converges, it has the limit L. By definition, $\exists N \in \mathbb{P}$ and $N \geq 2$ such that

$$\begin{aligned} &|a_n - L| < 1 \\ \implies &L - 1 < a_n < L + 1 \quad \forall n \geq N. \end{aligned} \tag{1}$$

Write $M := \max\{|a_1|, \dots, |a_{N-1}|\}$. Then, as $-|a_j| \leq a_j \leq |a_j|\ \forall j$, we have

$$-M \leq a_j \leq M \text{ for } j = 1, \dots, N-1.$$

Combining this with (1), we have

$$\min\{-M, L-1\} \leq a_n \leq \max\{M, L+1\} \quad \forall n \in \mathbb{P}.$$

The set $\{a_n : n = 1, 2, 3, \dots\}$ is bounded above and bounded below. Hence $\{a_n\}$ is bounded.

2. Let $\{b_n\}$ be a sequence in $\mathbb{R}$ that converges to M such that $b_n \neq 0$ for every $n = 1, 2, 3, \dots$. Assume that $M \neq 0$. Prove that the sequence $\{1/b_n\}$ converges and that $\lim_{n\to\infty}(1/b_n) = 1/M$.

Hint. It may helpful to realise that

$$|M| - |b_n| \;\leq\; |b_n - M| \quad \forall n.$$

Use this appropriately when estimating $\big|(1/b_n) - (1/M)\big|$.

3. In this problem, you may assume the following **without** proof:

- (i) For each positive real a and $n \in \mathbb{N} - \{0\}$, there exists a **unique** positive solution of the equation $x^n = a$. Denote this number as $a^{1/n}$. (The existence of $a^{1/n}$ is a consequence of the l.u.b. property of $\mathbb{R}$.)
- (ii) For $m, n \in \mathbb{N} - \{0\}$ and for any $x \in \mathbb{R}$,

$$(x^m)^n \;=\; (x^n)^m \;=\; x^{mn}.$$

Recall that any positive rational number q is of the form m/n, where $m, n \in \mathbb{N} - \{0\}$. Now, for any real $a > 0$, we define

$$a^q \;:=\; (a^m)^{1/n}. \tag{2}$$

(a) Show that a^q, as given by (2), is well-defined: i.e., if $q = \mu/\nu$, $\mu, \nu \in \mathbb{N} - \{0\}$ is a different representation of q, then $(a^m)^{1/n} = (a^\mu)^{1/\nu}$.

(b) Having defined what a^q means, $a > 0$ and q a positive rational, prove the following: *Let q be a positive rational. Then, the sequence $\{1/n^q\}$ converges and*

$$\lim_{n\to\infty} \frac{1}{n^q} = 0.$$

Sketch of solution: We shall only prove (a) in detail.

a) Let μ/γ and m/n be two representations of q, where $m, n, \mu, \gamma \in \mathbb{N} - \{0\}$. So,

$$m\gamma = \mu n =: c$$

Now write $x := (a^m)^{1/n},\ y := (a^\mu)^{1/\gamma}$. Then, by definition,

$$x^n = a^m \text{ and } y^\gamma = a^\mu. \tag{3}$$

Therefore, by (3), we get:

$$x^{n\mu} = (x^n)^\mu = (a^m)^\mu = a^{m\mu} = (a^\mu)^m = (y^\gamma)^m = y^{\gamma m}.$$

From this, we have $x^c - y^c = 0$. By an identity from high-school algebra,

$$(x - y)\left(\sum_{j=0}^{c-1} x^j y^{c-1-j}\right) = 0.$$

Since $x, y > 0$, the above equation implies $(x - y) = 0$. By the definitions of x and y,

$$(a^m)^{1/n} = (a^\mu)^{1/\gamma}.$$

b) The proof of part (b) is very similar to the proof, presented in class, that $\lim_{n\to\infty} 1/\sqrt{n} = 0$, **except for one** major additional requirement. This requirement is met by the following

Lemma. *Let x, y be positive real numbers. Then, for any $m \in \mathbb{N} - \{0\}$, $y > x$ if and only if $y^m > x^m$.*

Proof of lemma. Since there is nothing to prove if $m = 1$, fix $m \in \mathbb{N} - \{0, 1\}$. Then, given that $x, y > 0$, we have

$$\begin{aligned} y^m - x^m = (x - y)\left(\sum_{j=0}^{m-1} x^j y^{m-1-j}\right) > 0 &\iff y - x > 0 \\ &\iff y > x, \end{aligned}$$

which establishes the lemma.

The remainder of the solution of part (b) is left to you to complete.

4. Does the sequence $\{a_n\}$, where

$$a_n = \frac{1 - (-1)^n}{2}, \quad n = 1, 2, 3, \ldots,$$

converge? If so, then what is its limit? **Justify** your answer.

5. In each case below, show that the series $\sum_{n=1}^\infty a_n$ converges, and find the sum:

a) $a_n = 1/(2n-1)(2n+1)$

b) $a_n = 1/(n^2-1)$

c) $a_n = n/(n+1)(n+2)(n+3)$

d) $a_n = (\sqrt{n+1}-\sqrt{n})/\sqrt{n^2+n}$

Sketch of solution: We make a **correction to part (*b*)** (which was conveyed via Teams): namely, for part (b), consider the series $\sum_{n=2}^{\infty} a_n$.

We shall see a solution of part (a). The solutions of (b) and (c) will follow from similar partial-fraction arguments.

a) Let us write, if possible,

$$\frac{1}{(2x-1)(2x+1)} = \frac{A}{2x-1} + \frac{B}{2x+1} \quad \forall x \in \mathbb{R} - \{-1/2, 1/2\}. \tag{4}$$

Note that

$$\begin{aligned}(4) \text{ is true } &\iff A(2x+1) + B(2x-1) = 1 \quad \forall x \in \mathbb{R} \\ &\iff \begin{cases} 2A + 2B = 0 \\ A - B = 1, \end{cases}\end{aligned}$$

by high-school algebra. The linear equation has the unique solution $(A, B) = (1/2, -1/2)$. It follows that

$$a_n = \frac{1/2}{2n-1} - \frac{1/2}{2n+1} = \frac{1/2}{2n-1} - \frac{1/2}{2(n+1)-1}, \quad n = 1, 2, 3, \ldots$$

Thus, the b_n appearing in the convergence theorem for telescoping series is

$$b_n = \frac{1/2}{2n-1} = \frac{1/2n}{2-(1/n)}, \quad n = 1, 2, 3, \ldots$$

Since $\lim_{n\to\infty}(1/n) = 0$, the denominator of the right-hand quotient above has the limit $2 \neq 0$. Thus, by the limit theorem for quotients

$$\lim_{n\to\infty} b_n = 0.$$

Hence, by the convergence theorem

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = b_1 - \lim_{n\to\infty} b_n = \frac{1}{2}.$$

d) Observe that

$$\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}, \quad n = 1, 2, 3, \ldots$$

Thus, the given series is clearly telescoping. Now argue as in the last few sentences of the solution to part (a) to conclude:

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = 1.$$