UMA 101 : ANALYSIS & LINEAR ALGEBRA – I
AUTUMN 2023

HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 5 PROBLEMS

Instructor: GAUTAM BHARALI **Assigned: SEPTEMBER 5, 2023**

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1. Let $\{a_n\}$ be a convergent sequence with limit L. Prove that the sequence $\{b_n\}$, where

$$b_n = \frac{a_1 + \cdots + a_n}{n},$$

converges to L.

Sketch of solution: Since $\{a_n\}$ converges to L, it is a bounded sequence. Thus, $\exists c > 0$ such that $|a_n| \leq c \; \forall n$. Then,

$$|a_n - L| \leq |a_n| + |L| \leq c + |L| \quad \forall n = 1, 2, 3, \dots \tag{1}$$

Fix $\epsilon > 0$. Then, $\exists N_1 \in \mathbb{P}, N_1 \geq 2$, such that

$$|a_n - L| < \epsilon/2 \quad \forall n \geq N_1.$$

By the triangle inequality, we have

$$\begin{aligned}\left|\frac{a_1 + \cdots + a_n}{n} - L\right| &= \left|\frac{(a_1 - L) + \cdots + (a_n - L)}{n}\right| \\ &\leq \sum_{j=1}^{n} \frac{|a_j - L|}{n}\end{aligned}$$

From the above inequality and (1), we get

$$\left|\frac{a_1 + \cdots + a_n}{n} - L\right| \leq \frac{(N_1 - 1)(c + |L|)}{n} + \frac{n - N_1 + 1}{n}\left(\frac{\epsilon}{2}\right) \quad \forall n \geq N_1 \tag{2}$$

Let us examine the first term on the right-hand side of (2). From theorems about limits presented in class, we know that

$$\lim_{n\to\infty} \frac{(N_1 - 1)(c + |L|)}{n} = 0.$$

Thus, $\exists N_2 \in \mathbb{P}$ such that $0 < (N_1 - 1)(c + |L|)/n < \epsilon/2 \; \forall n \geq N_2$. Set $N := \max(N_1, N_2)$. Combining the previous inequality with (2), we have

$$\left|\frac{a_1 + \cdots + a_n}{n} - L\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} \; \forall n \geq N.$$

Since $\epsilon > 0$ in the above argument was arbitrary, the desired result follows.

2. Let $\sum_{n=1}^{\infty} a_n$ be a convergent real series. Prove that $\lim_{n\to\infty} a_n = 0$.

Hint. Apply the lemma discussed during the September 4 lecture to an appropriate sequence.

3. Determine whether or not each of the following non-negative series converges. Give **justifications.**

(a) (Apostol, 10.14, Prob. 1) $\sum_{n=1}^{\infty} n/(4n-3)(4n-1)$

(b) $\sum_{n=1}^{\infty} |\sin(5n^2)|/n^2$

(c) $\sum_{n=1}^{\infty} \big(3+(-1)^n\big)/3^n$

(d) (Apostol, 10.14, Prob. 7) $\sum_{n=1}^{\infty} n!/(n+2)!$

(e) $\sum_{n=1}^{\infty} b_n/5^n$, where $\{b_n\}$ is a bounded sequence with non-negative terms

(f) $\sum_{n=1}^{\infty}(n^2+(-1)^n)/n^2$

Sketch of solution: We shall prove parts (b), (d), and (f).

(b) We note that

$$\frac{|\sin(5n^2)|}{n^2} \leq \frac{1}{n^2} \quad \forall n = 1, 2, 3, \ldots$$

By the p-series test, $\sum_{n=1}^{\infty}(1/n^2)$ converges. So, by the comparison test, the given series converges.

(d) We simplify

$$\frac{n!}{(n+2)!} = \frac{1}{(n+1)(n+2)} =: a_n.$$

We first intuit that a_n behaves like $1/n^2$ for all n large enough. Thus, we should find a series $\sum_{n=1}^{\infty} b_n$ that is convergent and such that, for some N

$$0 \leq \frac{1}{(n+1)(n+2)} \leq b_n \quad \forall n \geq N.$$

Now, note that

$$\frac{1}{(n+1)(n+2)} \leq \frac{1}{n^2} \quad \forall n.$$

As $\sum_{n=1}^{\infty}(1/n^2)$ is convergent, by the p-series test, by the comparison test the given series converges.

Alternatively, we may notice that the series is a telescoping series. By the same approach as sketched in the discussion on Assignment 4, we deduce that the given series converges.

f) Use the divergence test appropriately to show that the series diverges.

4. State whether or not each of the following non-negative series converges. Give **justifications.**

a) (Apostol, 10.16, Prob. 13) $\sum_{n=1}^{\infty} \frac{n^3\left(\sqrt{2}+(-1)^n\right)^n}{3^n}$

b) $\sum_{n=1}^{\infty}(n!)^2/(2n)!$

Note. You must use **only** the tests and results discussed in class or assigned for self-study.

Sketch of solution: We will discuss part (a).

Write $a_n := n^3(\sqrt{2}+(-1)^n)^n/3^n$. The series is neither telescoping nor geometric. It appears like the ratio test would be applicable, but the preconditions of the ratio test (in the form that we have studied) not apply to the given series. However, note that

$$0 \leq a_n \leq n^3\left(\frac{\sqrt{2}+1}{3}\right)^n \quad \forall n = 1, 2, 3, \ldots \tag{3}$$

Write $b_n := n^3(\sqrt{2}+1)^n/3^n$ and note that

$$\frac{b_{n+1}}{b_n} = \frac{(n+1)^3}{n^3}\left(\frac{\sqrt{2}+1}{3}\right) = \left(1+\frac{1}{n}\right)^3\left(\frac{\sqrt{2}+1}{3}\right) \quad \forall n = 1, 2, 3, \ldots$$

By theorems on the algebraic combinations of convergent sequences, we have

$$\lim_{n\to\infty} \frac{(n+1)^3}{n^3}\left(\frac{\sqrt{2}+1}{3}\right) = \frac{\sqrt{2}+1}{3} < 1.$$

By the ratio test, $\sum_{n=1}^{\infty} b_n$ converges. Then, by (3), the comparison test implies that the given series converges.