UMA 101 : ANALYSIS & LINEAR ALGEBRA – I
AUTUMN 2023

HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 9 PROBLEMS

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1. Let $\mathbb{F}$ be an ordered field.

a) Propose a definition for the "greatest lower-bound property" of $\mathbb{F}$.

b) For any set $S \subseteq \mathbb{F}$ that has a greatest lower bound, let $\inf S$ denote its greatest lower bound (this presupposes a fact discussed in class: i.e., if $S \subseteq \mathbb{F}$ has a greatest lower bound, then it is unique; you may freely use this fact **without proof**). Now let $A \subseteq \mathbb{F}$ be a non-empty set such that $\sup A$ exists. Define

$$-A := \{-x \in \mathbb{F} : x \in A\}.$$

Prove that $\inf(-A) = -\sup A$.

c) Show that if $\mathbb{F}$ has the least upper-bound property, then it has the greatest lower-bound property.

2. Consider the following result presented in class:

> **Theorem.** *Let I be a closed and bounded interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be a continuous function. Then f is bounded.*

Observe that there are three separate conditions on the pair (I, f) stated in the above theorem. Give examples showing that whenever any one of the above conditions is violated, even though the **other two** hold true, the conclusion of the above theorem does not follow.

Sketch of solution: First, let us consider an example where I has the conditions stated and $f : I \to \mathbb{R}$ is not continuous. Let:

$$I_1 := [0, 1] \quad \text{and} \quad f_1(x) := \begin{cases} 0, & \text{if } x = 0, \\ 1/x, & \text{if } x \in (0, 1]. \end{cases}$$

Note that as $1/n \in (0, 1]\ \forall n \in \mathbb{P}$,

$$\text{range}(f_1) \supseteq \{f_1(1/n) : n \in \mathbb{P}\} = \mathbb{P}.$$

We had proved that $\mathbb{P}$ is not bounded above. Thus $\text{range}(f_1)$ is not bounded.

We adapt the previous example, where I is not open, but all other conditions on the pair (I, f) hold true. Let:

$$I_2 := (0, 1] \quad \text{and} \quad f_2(x) := 1/x.$$

For similar reasons as above, f_2 is not bounded.

Finally, let I be closed but unbounded, and consider an example where $f : I \to \mathbb{R}$ is continuous:

$$I_3 = [0, +\infty) \quad \text{and} \quad f_3(x) := x.$$

As f_3 is the identity function on I_3 and I_3 is unbounded, range(f_3) is not bounded.

3. Let $a, b \in \mathbb{R}$, $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Show that $\mathsf{range}(f)$ is a closed interval.

4. Let $f(x) = x^r$, $x \geq 0$, where r is a positive *rational* number. (Refer to Problem 3 in Homework No. 4 to recall what x^r, $x \geq 0$, means for r a positive rational number.) For the moment, **assume without proof** that the functions $g_n(x) = x^{1/n}$, $x \geq 0$, $n = 1, 2, 3, \ldots$, are differentiable on $(0, \infty)$.**Without using the Chain Rule**, show that f is differentiable on $(0, \infty)$, and compute $f'(x)$.

Hint. You will have to use induction.

Sketch of solution: Note that the part of the problem involving computing $f'(x)$ was withdrawn. (This can be done by following up on the hint to use induction, but is laborious.) Recall that if $r > 0$ and $r \in \mathbb{Q}$, then

- r is of the form $r = m/n, m, n \in \mathbb{N} - \{0\}$, and
- For each fixed $x > 0, x^r := (x^m)^{1/n}$ and the number on the R.H.S. does not depend on the choice of the pair (m, n).

Now, this insight will suffice to prove that f is differentiable on $(0, \infty)$ using the Chain Rule (see Problem 5), but will not help in the present case! Using the ideas used in proving Problem 3 in Homework No.4, **establish the following** substitute for the second bullet-point above:

$(*)$ $x^r := (x^{1/n})^m$ and the number on the R.H.S does not depend on the choice of the pair (m, n).

Now, fix $m, n \in \mathbb{N} - \{0\}$ such that $r = m/n$. Now, fix $x > 0$. Use mathematical induction and the product rule to prove that if $\Sigma(k)$ is the statement below:

$$\Sigma(k) : \quad \underbrace{g_n \times g_n \times \cdots \times g_n}_{(k+1) \text{ factors}} \text{ is differentiable at } x,$$

$k \in \mathbb{N}$, then $\Sigma(k)$ is true $\forall k \in \mathbb{N}$. Then, $\Sigma(m)$ is the (true) statement that f is differentiable at x. Since $x > 0$ was arbitrary, f is differentiable on $(0, \infty)$.

5. Consider the function f in Problem 4. Establish the fact stated in Problem 4 using the Chain Rule this time.

6–9. For the functions given in Problems 19–22 of Section 4.6 of Apostol, **justify** why these functions are differentiable on $(0, \infty)$. Then, compute formulas for their derivatives.

Sketch of solution: These problems follow easily from the theorem on the differentiability of algebraic combinations of differentiable functions and from the outcome of Problem 4 (or Problem 5).

10. Let $f(x) = x^2 + ax + b$, $x \in \mathbb{R}$. Find all possible values of a and b for which the following holds true: the line $y = 2x$ is tangent to the graph of f at the point $(2, 4)$.

Sketch of solution: We first compute

$$f'(x) = 2x + a.$$

From the geometric interpretation of $f'(p)$ in terms of the tangent to graph(f) at the point $(p, f(p))$, $p \in \mathbb{R}$, (a, b) must satisfy the equations.

$$\begin{aligned} 2x + a\,\big|_{x=2} &= 2, \\ x^2 + ax + b\,\big|_{x=2} &= 4. \end{aligned}$$

This gives us $a = -2$ and $b = 4$.

FOR SELF-STUDY:

Work through as many of the problems in the range 20–29, Section 4.12 of Apostol, as you can.