

UMA 101 : ANALYSIS & LINEAR ALGEBRA – I
AUTUMN 2023
HOMEWORK 11

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Assigned: OCTOBER 31, 2023

1. Let $a < b$ be real numbers and let $s : [a, b] \rightarrow \mathbb{R}$ be a step function.

- a) Prove that s is integrable according to the abstract definition given in terms of upper and the lower integrals (i.e., that s is *Riemann integrable*).
- b) You have been given a **formula** for the integral of a step function on $[a, b]$. Show that the value of the integral of s given by the above-mentioned definition agrees with that given by the formula.

2. Let $a < b$ be real numbers and let $s : [a, b] \rightarrow \mathbb{R}$ be a step function. Let $c \in \mathbb{R}$. Show that

$$\int_a^b s(x) dx = \int_{a+c}^{b+c} s(x-c) dx.$$

Tip. In this case, it is clearly most efficient to work with the **formula** defining the integral of a step function.

3. Let f be a function defined on an interval $[-A, A]$, $A > 0$, and suppose $f|_{[0, A]}$ is Riemann integrable. Suppose f is an even function (i.e., $f(x) = f(-x)$ for any $x \in [-A, A]$). Prove that f is integrable and show that

$$\int_{-A}^A f(x) dx = 2 \int_0^A f(x) dx.$$

4. Fix $r > 0$ and define the non-negative function $f : [-r, r] \rightarrow \mathbb{R}$ as follows:

$$f(x) := \sqrt{r^2 - x^2}, \quad -r \leq x \leq r.$$

Assuming that $f \in \mathcal{R}[-r, r]$, what do you **expect** the value of $\int_{-r}^r f(x) dx$ to be? You are not being asked to provide a calculation or a rigorous argument; guess the expected answer and give a reason for this guess based on the motivation for the Riemann integral.

5. Let $a < b$ be real numbers and let $f : [a, b] \rightarrow \mathbb{R}$ be Riemann integrable on $[a, b]$. Show that for any $c, d \in \mathbb{R}$ such that $a \leq c < d \leq b$, $f|_{[c, d]}$ is Riemann integrable on $[c, d]$.