UMA 101 : ANALYSIS & LINEAR ALGEBRA – I

AUTUMN 2023

HOMEWORK 12

Instructor: GAUTAM BHARALI **Assigned: NOVEMBER 7, 2023**

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1. Let $a < b \in \mathbb{R}$ and let $f \in \mathscr{R}[a,b]$ be a step function. Let $c \in (a,b)$. Show that

$$f|_{[a,c]} \in \mathscr{R}[a,c] \quad \text{and } f|_{[c,b]} \in \mathscr{R}[c,b],$$

and that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Note. The first part of this problem is already established by Problem 5 of Homework 11.

2. Self-study. *Read* the statement of the Small-span Theorem (i.e., THEOREM 3.13) in Apostol's book. Next, study the proof of the fact that if $f : [a,b] \to \mathbb{R}$ is continuous, then f is Riemann integrable on $[a,b]$ (i.e., THEOREM 3.14 in Apostol's book).

3. Let $a < b \in \mathbb{R}$. Use the fact that if a function $f : [a,b] \to \mathbb{R}$ is continuous, then it is **uniformly** continuous, to give a short proof of the Small-span Theorem.

4. Let $a < b \in \mathbb{R}$ and let $f : [a,b] \to \mathbb{R}$ be a bounded function. The following discussion shows why $\underline{I}(f)$ and $\overline{I}(f)$ are called the "lower integral" and the "upper integral", respectively, of f.

a) Show that for any step function $s_1 : [a,b] \to \mathbb{R}$ such that $s_1 \leq f$ and any step function $s_2 : [a,b] \to \mathbb{R}$ such that $s_2 \geq f$,

$$\int_a^b s_1(x)\,dx \leq \int_a^b s_2(x)\,dx.$$

b) Now deduce that $\underline{I}(f) \leq \overline{I}(f)$.

5. Show that the function $f_n : \mathbb{R} \to \mathbb{R}$, given by $f_n(x) := x^n$, is not uniformly continuous for $n \in \mathbb{N} - \{0,1\}$.

6. You are given a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous and satisfies

$$\int_0^x f(t)dt = 1 + x^2 + x\sin(2x) \quad \forall x \in \mathbb{R}.$$

Compute $f(\pi/4)$.

7–8. Solve Problems 17 and 22 from Section 5.5 of Apostol.

9. Recall the definition of the *natural logarithm* $\log : (0,\infty) \to (0,\infty)$ introduced in class.

a) Prove that log is strictly increasing.

b) Assume **without** proof that the range of log is $\mathbb{R}$. Thus, $E := \log^{-1}$ is a function defined on $\mathbb{R}$. E is called the *exponential function*; recall that we frequently write $e^x := E(x)$ for $x \in \mathbb{R}$. With this notation, prove that

$$e^x\, e^y \;=\; e^{x+y} \quad \forall x, y \in \mathbb{R}.$$

10. Based on our discussion on the Leibnizian notation and the meaning of the left-hand side below, **justify** the equation:

$$\int \frac{1}{x}\, dx \;=\; \log|x| + C.$$