UMA 101 : ANALYSIS & LINEAR ALGEBRA – I
AUTUMN 2023
HOMEWORK 1

Instructor: GAUTAM BHARALI **Assigned: AUGUST 8, 2023**

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1. In class we encountered one of the axioms of Set Theory, stated as "The rule defining when two sets are equal," and for which you were referred to **Section I-2.2** of Apostol's book. Using this rule, justify the following equalities of sets:

(a) $\{a, a\} = \{a\}$

(b) $\{a, b\} = \{b, a\}$

(c) $\{a\} = \{b, c\}$ if and only if $a = b = c$.

2. (Prob. 20(b) from Apostol, Section I-2.5) Show that one of the two expressions below is always right and that the other is sometimes wrong:

$$i)\ A - (B - C) = (A - B) \cup C,$$
$$ii)\ A - (B \cup C) = (A - B) - C.$$

(**Note.** What this means is that you must provide a proof of the expression that you think is always right, and you must provide one counterexample showing that the other is not.)

3. Consider the following axiom of Set Theory, which has been referred to in class:
(The Set-building Axiom) *Let A be a set and, for each $x \in A$, let $P(x)$ denote a statement involving x. Then the collection described as:*

$$\{x \in A : P(x) \text{ is true}\}$$

is a set.

Now let $\mathscr{C}$ be a non-empty set of sets. By appealing to the Set-building Axiom, explain why, unlike $\cup_{A\in\mathscr{C}} A$ — which requires the Axiom of Union to declare it to be a set — the collection $\cap_{A\in\mathscr{C}} A$ does not require a separate "Axiom of Intersection" for one to know that it is a set.

4. Prove the De Morgan law whose proof was **not** given in class. Namely, if X is a set and $\mathscr{C}$ is a non-null set whose elements are subsets of X, then show that

$$X - \left(\bigcap_{A\in\mathscr{C}} A\right) = \bigcup_{A\in\mathscr{C}} (X - A).$$

5. For self-study: A *field* $\mathbb{F}$ is a set equipped with a pair of rules denoted by "+" (called the *sum*) and "$\cdot$" (called the *product*) defined for every pair of elements $x, y \in \mathbb{F}$ that obey

Axiom 0. Closure laws:

$$x + y \in \mathbb{F} \quad \text{and} \quad x \cdot y \in \mathbb{F} \quad \forall x, y \in \mathbb{F},$$

and **six** other axioms. These are AXIOMS 1–6 in Apostol, **Section I-3.2,** with the understanding that x, y, z, etc., are elements in $\mathbb{F}$ (instead of just $\mathbb{R}$ as given in Section I-3.2). Please **read the statements** of each of these six axioms.

The following problem will go a little beyond what has been taught until today (but the relevant material **will** be taught this week).

6. Refer to Peano's Axioms. For a natural number n, $S(n)$ will denote the successor of n. Let "+" denote the Peano addition between two natural numbers (which formalises the addition you learnt as children). Define:

$$\begin{aligned} 1 &:= S(0), \\ 2 &:= S(1) = S(S(0)), \\ 3 &:= S(2) = S(S(1)) = S\big(S(S(0))\big). \end{aligned}$$

Using the rules of Peano addition, justify that

(*a*) $1 + 1 = 2$.

(*b*) $1 + 2 = 3$.

Note. You may freely use the fact $n + m = m + n$ for all $m, n \in \mathbb{N}$ **without proof.** Using this will provide a *somewhat* shorter proof of (*b*) than the one resulting from following the rules of Peano addition slavishly.