UMA 101 : ANALYSIS & LINEAR ALGEBRA – I

AUTUMN 2023

HOMEWORK 4

Instructor: GAUTAM BHARALI **Assigned: AUGUST 29, 2023**

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1. Let $\{a_n\}$ be a sequence in $\mathbb{R}$. We say "$\{a_n\}$ is bounded" if the **set** $\{a_n : n = 1, 2, 3, \dots\}$ is bounded above and bounded below. Prove that if $\{a_n\}$ converges, then it is bounded.

Tip. If $\{c_1, c_2, \dots, c_N\} \subset \mathbb{R}$ is a finite set, then you may freely use $\max\{c_1, c_2, \dots, c_N\}$ — if required — without spelling out its definition (which states exactly what you understood by it in school) or proving that $\max\{c_1, c_2, \dots, c_N\}$ exists.

2. Let $\{b_n\}$ be a sequence in $\mathbb{R}$ that converges to M such that $b_n \neq 0$ for every $n = 1, 2, 3, \dots$. Assume that $M \neq 0$. Prove that the sequence $\{1/b_n\}$ converges and that $\lim_{n\to\infty}(1/b_n) = 1/M$.

Hint. It may helpful to realise that

$$|M| - |b_n| \leq |b_n - M| \quad \forall n.$$

Use this appropriately when estimating $\big|(1/b_n) - (1/M)\big|$.

3. In this problem, you may assume the following **without** proof:

(*i*) For each positive real a and $n \in \mathbb{N} - \{0\}$, there exists a **unique** positive solution of the equation $x^n = a$. Denote this number as $a^{1/n}$. (The existence of $a^{1/n}$ is a consequence of the l.u.b. property of $\mathbb{R}$.)

(*ii*) For $m, n \in \mathbb{N} - \{0\}$ and for any $x \in \mathbb{R}$,

$$(x^m)^n = (x^n)^m = x^{mn}.$$

Recall that any positive rational number q is of the form m/n, where $m, n \in \mathbb{N} - \{0\}$. Now, for any real $a > 0$, we define

$$a^q := (a^m)^{1/n}. \tag{1}$$

(*a*) Show that a^q, as given by (1), is well-defined: i.e., if $q = \mu/\nu$, $\mu, \nu \in \mathbb{N} - \{0\}$ is a different representation of q, then $(a^m)^{1/n} = (a^\mu)^{1/\nu}$.

(*b*) Having defined what a^q means, $a > 0$ and q a positive rational, prove the following: *Let q be a positive rational. Then, the sequence $\{1/n^q\}$ converges and*

$$\lim_{n\to\infty} \frac{1}{n^q} = 0.$$

4. Does the sequence $\{a_n\}$, where

$$a_n = \frac{1 - (-1)^n}{2}, \quad n = 1, 2, 3, \dots,$$

converge? If so, then what is its limit? **Justify** your answer.

P.T.O.

5. In each case below, show that the series $\sum_{n=1}^{\infty} a_n$ converges, and find the sum:

a) $a_n = 1/(2n-1)(2n+1)$

b) $a_n = 1/(n^2-1)$

c) $a_n = n/(n+1)(n+2)(n+3)$

d) $a_n = (\sqrt{n+1} - \sqrt{n})/\sqrt{n^2+n}$