UMA 101 : ANALYSIS & LINEAR ALGEBRA – I

AUTUMN 2023

HOMEWORK 6

Instructor: GAUTAM BHARALI **Assigned: SEPTEMBER 12, 2023**

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1. Fix some positive integer N. Show that the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the series $\sum_{n=N}^{\infty} a_n$ is convergent.

2. Let p be a real number contained in an open interval I. Let f be a $\mathbb{R}$-valued function such that $f(x)$ is defined at each $x \in I$ except perhaps at $x = p$. Let $A \in \mathbb{R}$. How do you express quantitatively (involving parameters like ε, etc., in an appropriate way) the statement, "$f(x)$ does **not** have the limit A as x approaches p"?

3. Let p be a real number contained in an open interval I. Let f, g be $\mathbb{R}$-valued functions such that $f(x)$ and $g(x)$ are defined at each $x \in I$ except perhaps at $x = p$. Suppose $\lim_{x\to p} f(x) = A$ and $\lim_{x\to p} g(x) = B$. Prove using the "ε-δ definition" that

$$\lim_{x\to p} f(x)g(x) \;=\; AB$$

directly **without** first assuming — as has been done in the textbook — that either A or B equals 0

4. Show that

$$\lim_{x\to 0} \frac{\sin(6x) - \sin(5x)}{x}$$

exists. Give **justifications** in terms of the limit theorems that are used.

Note. You may use standard trigonometric identities learnt in high school **without** deriving them.

5. Let n be some (fixed) positive integer and let $p \in \mathbb{R}$. Complete the following outline to show that $\lim_{x\to p} x^n = p^n$ using **only** the "ε-δ definition" (i.e., **without** using the limit theorem stated in Problem 3 above):

(a) Establish the desired limit for the case $n = 1$ using the "ε-δ definition".

(b) Now, use Part (a) appropriately to establish the stated limit.

6. Show, using any of the theorems on the algebra of limits, that the limit

$$\lim_{x\to 0} \frac{1 - \sqrt{1 - x^2}}{x^2}$$

exists.