UMA 101 : ANALYSIS & LINEAR ALGEBRA – I

AUTUMN 2023

HOMEWORK 8

Instructor: GAUTAM BHARALI **Assigned: OCTOBER 10, 2023**

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1. Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, and let $p \in I$. Let $\{a_n\} \subset I$ be a sequence such that $\lim_{n\to\infty} a_n = p$. Suppose f is continuous at p. Then, prove that $\{f(a_n)\}$ is a convergent sequence and $\lim_{n\to\infty} f(a_n) = f(p)$.

Remark. The above result provides yet another method of constructing new convergent sequences from known convergent sequences.

2. Fix a number $p \in \mathbb{R}$. Let θ denote an arbitrary real number. We showed in class that

$$|\sin(\theta + p) - \sin(p)| \;\leq\; |\sin\theta| + 2\sin^2\left(\frac{\theta}{2}\right).$$

From this, deduce that the sine function is continuous at p. You may freely use **without proof** the fact that $|\sin\theta| \leq |\theta|\ \forall\theta \in \mathbb{R}$ (the easiest proof of which you know from Euclidean geometry).

3–6. Solve Problems 7–10 in Section 3.8 of Apostol's book.

7. Let $a < b < c \in \mathbb{R}$. Suppose $g : [a, b] \to \mathbb{R}$ and $h : [b, c] \to \mathbb{R}$ are two continuous functions. You are given that $g(b) = h(b)$. Define the function

$$f(x) \;=\; \begin{cases} g(x), & \text{if } a \leq x \leq b, \\ h(x), & \text{if } b \leq x \leq c. \end{cases}$$

Use the $\varepsilon - \delta$ definition of continuity to show that f is continuous on $[a, c]$.

Note. From the **sequential** definition of continuity, it is almost immediate that f is continuous! The aim of this problem is to get you to work with the $\varepsilon - \delta$ definition.

8. Show that the equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \;=\; 0,$$

where $a_0, a_1, \ldots, a_{n-1}$ are real numbers, has at least one root in $\mathbb{R}$ if n is odd.

9. Show that the equation $\sin x = x - 1$ has at least one real solution.

10. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$f(x) \;=\; \begin{cases} \sin x, & \text{if } x \leq c, \\ ax + b, & \text{if } x > c, \end{cases}$$

where a, b, c are real constants. Suppose a and b are fixed. Find *all* possible values of c such that f is continuous at $x = c$. You may use any result in this assignment sheet that may be relevant to solving this problem.