UMA 101 : ANALYSIS & LINEAR ALGEBRA – I
AUTUMN 2023
HOMEWORK 9

Instructor: GAUTAM BHARALI **Assigned: OCTOBER 18, 2023**

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1. Let $\mathbb{F}$ be an ordered field.

a) Propose a definition for the "greatest lower-bound property" of $\mathbb{F}$.

b) For any set $S \subseteq \mathbb{F}$ that has a greatest lower bound, let $\inf S$ denote its greatest lower bound (this presupposes a fact discussed in class: i.e., if $S \subseteq \mathbb{F}$ has a greatest lower bound, then it is unique; you may freely use this fact **without proof**). Now let $A \subseteq \mathbb{F}$ be a non-empty set such that $\sup A$ exists. Define

$$-A := \{-x \in \mathbb{F} : x \in A\}.$$

Prove that $\inf(-A) = -\sup A$.

c) Show that if $\mathbb{F}$ has the least upper-bound property, then it has the greatest lower-bound property.

2. Consider the following result presented in class:

Theorem. *Let I be a closed and bounded interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be a continuous function. Then f is bounded.*

Observe that there are three separate conditions on the pair (I, f) stated in the above theorem. Give examples showing that whenever any one of the above conditions is violated, even though the **other two** hold true, the conclusion of the above theorem does not follow.

3. Let $a, b \in \mathbb{R}$, $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Show that $\mathsf{range}(f)$ is a closed interval.

4. Let $f(x) = x^r,\ x \geq 0$, where r is a positive *rational* number. (Refer to Problem 3 in Homework No. 4 to recall what $x^r,\ x \geq 0$, means for r a positive rational number.) For the moment, **assume without proof** that the functions $g_n(x) = x^{1/n},\ x \geq 0,\ n = 1, 2, 3, \dots,$ are differentiable on $(0, \infty)$.**Without using the Chain Rule**, show that f is differentiable on $(0, \infty)$, and compute $f'(x)$.

Hint. You will have to use induction.

5. Consider the function f in Problem 4. Establish the fact stated in Problem 4 using the Chain Rule this time.

6–9. For the functions given in Problems 19–22 of Section 4.6 of Apostol, **justify** why these functions are differentiable on $(0, \infty)$. Then, compute formulas for their derivatives.

10. Let $f(x) = x^2 + ax + b,\ x \in \mathbb{R}$. Find all possible values of a and b for which the following holds true: the line $y = 2x$ is tangent to the graph of f at the point $(2, 4)$.

FOR SELF-STUDY:

Work through as many of the problems in the range 20–29, Section 4.12 of Apostol, as you can.