Algebraic topology in low-dimensional topology

Let M be closed, oriented, 4-dimensional manifold.
Assume that M is imply-connected.

$$H_1(M) = 0 \; ; \; H_3(M) = H^1(M) \cong \mathrm{Hom}(H_1(M), \mathbb{Z}) = 0$$

Perfect pairing (as $H_1(M) = 0$)

$$H^2(M) \times H_2(M) \longrightarrow \mathbb{Z}$$

$$\underset{H^2(M)}{\cong \uparrow [M] \cap -}$$

Thus, we have a symmetric, bilinear, unimodular form.

$$H^2(M) \times H^2(M) \longrightarrow \mathbb{Z}$$

$\mathbb{Z}^k \cong H^2(M) \rightsquigarrow \begin{bmatrix} & \end{bmatrix}_{k \times k}$ $\quad$ det $= \pm 1 \iff$ perfect pairing

E.g. $M = \mathbb{C}P^2$ etc., $H^2(M) = \mathbb{Z}$ $(= H_2(M))$

Then if $H^2(M) = [\varphi]$, then $(\varphi \cup \varphi)[M] = \pm 1$

In fact, 1 for $\mathbb{C}P^2$ and -1 for $\overline{\mathbb{C}P^2}$

Hence $f_* : H_2(M) \to H_2(M)$ determines $f_* : H_4(M) \circlearrowleft$

$(d_4 = d_2^2)$ or $(d_4 = -d_2^2)$

$\Rightarrow f$ has fixed points.

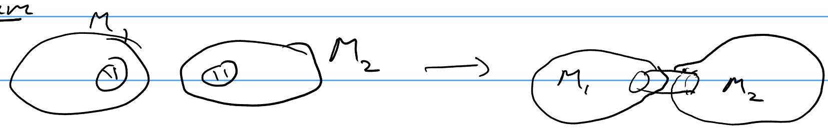
E.g. $S^2 \times S^2 \rightsquigarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = H$ - Even $q(x) \in 2\mathbb{Z}$ if $x \in \mathbb{Z}$
- Indefinite

Also get E_8 - definite

$K3 \longrightarrow 2E_8 \oplus 3H$

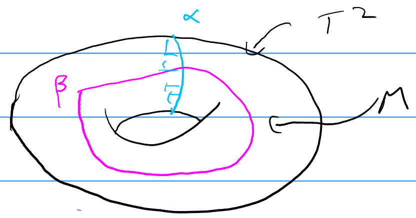
Connected sum

$M_1 \# M_2$



$\rightsquigarrow H^2(M_1) \oplus H^2(M_2)$

Question: Does the torus bound a contractible 3-manifold M



Suppose α bounds a surface Σ (as M is contractible, true)

Then $\beta \cdot \Sigma = \beta \cdot \alpha \neq 0$, so $[\beta] \neq 0$ in $H_1(M)$,

so M is not contractible.

More precisely:

$$\begin{array}{ccccc} \Sigma & \longmapsto & \alpha & \longmapsto & 0 \\ H_2(M,\partial M) & \longrightarrow & H_1(\partial M) & \longrightarrow & H_1(M) \end{array}$$

By Poincaré duality, $\exists \beta$ s.t. $\alpha \cdot \beta \neq 0$ ($(PD(\alpha) \cup PD(\beta))[T^2]$)

$$H_2(M, \partial M) \xrightarrow{PD} H^1(M) \; ; \quad \Sigma = [M] \cup \varphi$$

$[M] \cup$

Similarly, $\alpha = \overset{T}{[\partial M]} \cup \psi$

By naturality $\underset{H^1(M)}{\varphi}(\underset{H_1(M)}{[\beta]}) = \underset{H^1(\partial M)}{\psi}([\beta]) \neq 0$ (as $\alpha \cdot \beta \neq 0$)

Thus M is not contractible, contradiction

- The same argument with $\mathbb{Z}/2$ shows $\mathbb{R}P^2$ cannot bound M (not necessarily orientable)
  - $H_1(\mathbb{R}P^2; \mathbb{Z}/2) = \mathbb{Z}/2$

General question: Does $\pi_1(M)$ determine homotopy type?

Lens Spaces: L(p, q) ; $\gcd(p,q)=1$

$\mathbb{Z}/_p \curvearrowright S^3$ generated by $(z_1, z_2) \mapsto (z_1 e^{2\pi i/p}, z_2 e^{2\pi i q/p})$

'~~Obvious~~' Nice homeomorphism: $L(p,q) \underset{\substack{\uparrow \\ \text{flip } z_1 \,\&\, z_2 \\ (z_1,z_2)\mapsto(z_2,z_1)}}{=} L(p,q^{-1}) \overset{\text{orientation reversing}}{=} L(p,-q) = L(p,-q^{-1})$

$(z_1,z_2) \mapsto (z_1, \bar{z}_2)$

- $L(p,q_1) = L(p,q_2)$ if $q_1 \equiv q_2 \pmod p$
- q^{-1} means $q \cdot q^{-1} \equiv 1 \pmod p$

Bockstein and mod p pairing

$H_1(M) = \mathbb{Z}/_p$; $H^1(M;\mathbb{Z}/_p) = \mathbb{Z}/_p$

$0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \longrightarrow \mathbb{Z}/_p \to 0$ (of modules)

gives the s.e.s

$0 \to C^*(M,\mathbb{Z}) \xrightarrow{\times p} C^*(M,\mathbb{Z}) \longrightarrow C^*(M,\mathbb{Z}/_p) \to 0$

Hence the long exact sequence

$\to \underset{\substack{\shortparallel \\ 0}}{H^1(M,\mathbb{Z})} \to H^1(M;\mathbb{Z}/_p) \xrightarrow{\delta} \underset{\substack{\wr\shortparallel \\ \mathbb{Z}/p}}{H^2(M,\mathbb{Z})} \xrightarrow[=0]{\times p} H^2(M;\mathbb{Z}) \to H^2(M,\mathbb{Z}/_p)$

δ is the 'Bockstein' homomorphism

$H^1(M,\mathbb{Z}/_p) \times H^2(M,\mathbb{Z}/_p) \to \mathbb{Z}/_p$

$(\alpha, \beta) \mapsto (\alpha \cup \beta)[M]$

$H_1(M,\mathbb{Z}/_p)$ $\qquad$ $H_3(M;\mathbb{Z}/p)$

gives $H^1(M,\mathbb{Z}/_p) \times H^1(M,\mathbb{Z}/_p) \to \mathbb{Z}/_p$

which is homotopy-invariant.

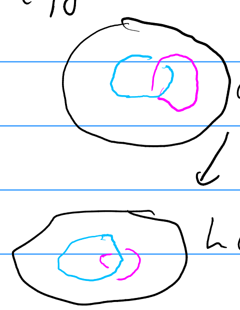
$f_* H_1 \xrightarrow{\times k} H_1$, $f_*: H^1 \xrightarrow{\times k} H^1$

$\simeq\downarrow\delta \quad \simeq\downarrow\delta$

$H^2 \longrightarrow H^2$

commuting

so $H^3 \to H^3$ by using the pairing



linking number well defined mod p.

'Approximate formula'

$f_* = z \mapsto kz$

then $\deg(f) \equiv k^2 \cdot q' \cdot q^{-1} \pmod p$

$\mathbb{Z}/p \cong \pi_1(L(p,q))$

Lemma: If $f: L(p,q) \to L(p,q')$, $f_*: H_1(L(p,q)) \to H_1(L(p,q'))$ determines the degree mod p of f, i.e. $f_*: H_3(L(p,q)) \to H_3(L(p,q'))$

- In particular, f can be a (orientation preserving) homotopy equivalence iff $\deg(f) \equiv \pm 1$ (or $\deg(f) \equiv 1$) (mod p)

Change degree by p: $L(p,q) = L(p,q) \# S^3$

$L(p,q')$ — Covering

Statement: $L(p,q) \underset{o.p}{\overset{h.e.}{\simeq}} L(p,q') \iff$

$\exists k$ s.t. $q' \equiv k^2 q \pmod p$

· $L(7,1) \overset{h.e.}{\simeq} L(7,2)$ as $2 \equiv 3^2 \cdot 1 \pmod 7$

Theorem (Reidemeister, Moise, Bonahon–Otal, Rubinstein–Scharlemann)

$L(7,1) \neq L(7,2)$ not homeomorphic

Milnor: Two complexes . . . .

Fact: Only homeomorphisms are the 'nice' ones.

· $L(8,1) \circlearrowleft$ has a homotopy equivalence not homotopic to a homeomorphism. ($k \mapsto 3k$)

Consequence: Right & left trefoil are different

$K \subset S^3$

M 2-fold branched cover

get $L(3,1)$ & $L(3,-1)$

$\pi_1(M) = \mathbb{Z}^3 = \pi_1(T^3)$, then $M \overset{h.e.}{\simeq} T^3$

· Consider $\tilde{M}$: $\pi_1(\tilde{M}) = 1 \Rightarrow H_1(\tilde{M}) = 0$

· $H_3(\tilde{M}) = 0$ as $\tilde{M}$ is non-compact (like fundamental class)

· $H_2(\tilde{M}) = H^1_c(\tilde{M})$; $H^1(\tilde{M}) = 0$

Fact: $H^1_c(\tilde{M}) = 0$ as ~~$\pi_1(M) =$~~ $\pi_1(M)$ has $\boxed{1 \text{ end}}$

$H_k(\tilde{M}) = 0 \ \forall k \geq 3$

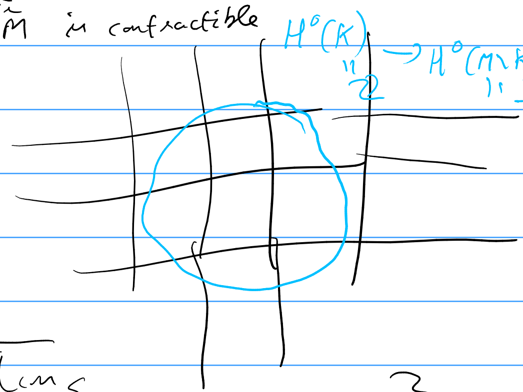
By Hurewicz: $\pi_k(\tilde{M})$ is trivial $\forall k$

$\Rightarrow \tilde{M}$ is contractible

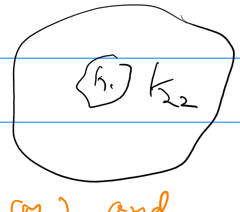
$H^1_c(\mathbb{R}) = \mathbb{Z}$

$H^1_c(M) = \varinjlim H^1(M, M \setminus K)$

$\underset{\mathbb{Z}}{H^0(K)} \to \underset{\mathbb{Z}}{H^0(M\setminus K)} \to \underset{=0}{H^1(M, M\setminus K)} \to \underset{0}{H^1(M)}$



Ends of X: $\varprojlim_{K \subset X \text{ compact}} \{\text{Components of } X \setminus K\}$



Qn: If M_1, M_2 are closed n-manifolds with $\pi_1(M_1) = \pi_1(M_2)$ and M_1, M_2 aspherical, are M_1, M_2 homeomorphic? (Borel Conjecture)