HEEGARD SPLITTINGS

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1. Introduction

The 3-sphere S^3 is the union of its northern hemisphere and its southern hemisphere. Thus, S^3 is the union of two balls, with their boundaries identified using an (orientation reversing) diffeomorphism. This is the simplest example of a so called *Heegard splitting*.

We cannot get many manifolds by gluing balls in this manner. Indeed S^3 is all we get.

Exercise 1. *Show that the only manifold obtained by gluing a pair of balls as above is S^3.*

Remark 1. The above exercise is a little more subtle than it may seem at first. In higher dimensions this construction gives exotic spheres.

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More generally, we shall glue so called *handlebodies* along their boundaries. We shall see that all closed 3-manifolds can be obtained in this manner. First, we look more closely at the simplest of these handlebodies to get our next example.

2. A basic example

2.1. **The solid torus.** The solid torus is the product $D^2 \times S^1$. It is useful to look at it from a couple of other points of view.

Firstly, one can obtain a solid torus from B^3 by *attaching a* 1*-handle*. Namely, to a pair of disjoint discs in $S^2 = \partial B^3$, glue the boundary $\{0,1\} \times D^2$ of $[0,1] \times D^2$, in such a way as to obtain an orientable manifold.

There is another way to obtain the solid torus from B^3. Take a diameter α in the solid torus. Delete the interior of a regular neighbourhood of this arc. It is easy to see that $B^3 \setminus int(\mathcal{N}(\alpha)) = D^2 \times S^1$.

More generally, we can take any properly-embedded*unknotted* arc $\alpha \subset B^3$, i.e., an arc α such that there exists an arc β embedded in $S^2 = \partial B^3$ and an embedded disc $E \subset B^3$ such that $\partial E = \alpha \cup \beta$. On deleting the interior of a neighbourhood of this arc, we get a solid torus.

Both these descriptions play a key role in what follows.

2.2. S^3 **once more.** We now get a more interesting example of a Heegard splitting. Namely, as before $S^3 = B_1 \cup B_2$, where the B_i are 3-balls glued together along their boundaries. Now let α be an unknotted properly embedded arc in B_1. Let $H_1 = B_1 \setminus int(\mathcal{N}(\alpha))$ and $H_2 = B_2 \cup \mathcal{N}(\alpha)$.

Since H_1 is the result of deleting an open neighbourhood of an unknotted arc from a 3-ball, it is a solid torus. On the other hand, H_2 is obtained by adding a 1-handle to a 3-ball, and is hence also a solid torus.

Thus, S^3 is the union of two solid tori, glued along their boundary. This is another example of a *Heegard splitting* of S^3. If one thinks of S^3 as the one-point compactification of $\mathbb{R}^3$, then H_1 is just the regular neighbourhood of an unknot, and H_2 is the closure of its complement.

It is often useful, in 3-manifold topology, to think of S^3 as the unit ball in $\mathbb{C}^2$. One can see the above decomposition form this point of view.

Exercise 2. *Let* $H_i \subset S^3 \subset \mathbb{C}^2$*,* $i = 1, 2$*, be given by* $H_i = \{(z_1, z_2) \in S^2 : |z_i|^2 \geq 1/2\}$. *Show that this is a decomposition of* S^3 *into solid tori.*

We can also glue solid tori together along other diffeomorphisms of their boundaries. One obtains in this manner $S^2 \times S^1$ as well as the *lens spaces*. We shall treat these in detail later.

3. Handlebodies

A handlebody of genus g is a manifold obtained by adding g 1-handles to a 3-ball. In other words, take a 3-ball with $2g$ disjoint discs on it, which we regard as g pairs of discs. To each pair, attach a copy of $[0,1] \times D^2$ along its boundary.

The manifold obtained does not depend on the discs chosen. If the attaching maps are orientation reversing, then the resulting handlebody is oriented. This is frequently called the handlebody of genus g.

A couple of alternative descriptions of handlebodies are useful. Firstly, as with the solid torus, one can obtain the orientable handlebody of genus g by deleting neighbourhoods of arcs from a 3-ball.

Namely, take a collection of properly embedded arcs $\alpha_1, 1 \leq i \leq g$ in B^3 that are unknotted and unlinked. This means that there are disjoint embedded arcs β_i in $S^2 = \partial B^3$ and disjoint discs $E_i \subset B^3$ such that $\partial E_i = \alpha_i \cup \beta_i$. We leave it as an exercise to see that this is the handlebody of genus g.

Exercise 3. *The manifold obtained above is an oriented handlebody of genus g.*

Another useful description of handlebodies is that they are 3-manifolds that are regular neighbourhoods of graphs (contained in the manifold). Namely, the solid torus $D^2 \times S^1$ is a regular neighbourhood of $\{0\} \times S^1$. More generally, we may construct a graph consisting of the arcs $[0,1] \times \{0\}$ in each 1-handle $[0,1] \times D^2$, and arcs joining the endpoints of all these arcs to a common base point in $int(B^3)$.

Exercise 4. *If M is a 3-manifold and $\Gamma \subset M$ is a graph such that M is a regular neighbourhood of Γ, then M is a handlebody.*

Exercise 5. *If H is a handlebody of genus g, then $\pi_1(H) = F_g$ is the free group on g generators.*

4. Heegard splittings

Given two orientable handlebodies W_1 and W_2 of the same genus g, and an orientation-reversing diffeomorphism $f : \partial W_1 \to \partial W_2$, we may construct a manifold M^3 as

$$M^3 = W_1 \coprod_{\partial W_1 = \partial W_2} W_2$$

We shall see that every orientable 3-manifold can be constructed in this manner. The decomposition of M into the handlebodies W_1 and W_2 is called a *Heegard splitting* of M. The surface $\partial W_1 = \partial W_2 \subset M$ is called a *Heegard surface.*

Exercise 6. *What happens if f is orientation-preserving?*

To construct non-orientable 3-manifolds, one glues non-orientable handlebodies of the same genus along their boundaries. A fundamental theorem asserts that these constructions give all 3-manifolds.

Theorem 2. *Every triangulated* 3*-manifold* M *has a Heegard splitting.*

Proof. Let M be a 3-manifold with a given triangulation T. The two handlebodies W_1 and W_2 in the Heegard splitting we construct will be regular neighbourhoods of the 1-skeleton T and the 1-skeleton of its dual triangulation T'.

Consider a regular neighbourhood of the 1-skeleton Γ, and call its boundary F. Then F separates M into two pieces. One of these is a regular neighbourhood of the 1-skeleton, and hence is a handlebody. By considering the intersection of F with each tetrahedron, it is easy to see that the other manifold obtained is a regular neighbourhood of the 1-skeleton of the dual triangulation, and hence is also a handlebody. Thus we have a Heegard splitting. □

Remark 3. By a theorem of Moise and Bing, all 3-manifolds have triangulations. Thus the above result holds for all 3-manifolds.

Example 4.1. (Heegard splittings of S^3) S^3 has Heegard splittings of all genera. Namely, we first express $S^3 = B_1 \cup B_2$. Take g properly embedded arcs $\alpha_1, 1 \leq i \leq g$ in B_1 that are unknotted and unlinked. Let $H_1 = B_1 \setminus \cup_{i=1}^{g} int(\mathcal{N}(\alpha_i))$ and $H_2 = B_2 \cup \cup_{i=1}^{g} \mathcal{N}(\alpha_i)$. Then H_1 and H_2 are handlebodies.

These Heegard splittings of S^3 are called the *standard* Heegard splittings of S^3. It is natural to ask whether there are any Heegard splittings of S^3 different from these.

The natural equivalence on Heegard splittings is *isotopy.* We say that two Heegard splittings of S^3 are isotopic if the corresponding Heegard surfaces are isotopic. A theorem of Waldhausen classifies Heegard splittings of S^3. For the proof, we refer to ?

Theorem 4 (Waldhausen)**.** *Any Heegard splitting of* S^3 *is isotopic to a standard one.*

Exercise 7. *Show that the standard Heegard splitting does not depend, up to isotopy, on the choice of the arcs* α_i*.*

5. Heegard diagrams

So far we do not have many interesting new examples of 3-manifolds. To efficiently construct and study manifolds using Heegard splittings, we shall need to introduce *Heegard diagrams.* A Heegard diagram is essentially a handle-decomposition of a 3-manifold.

5.1. Handlebodies once more.

We first need yet another way to construct handlebodies. This is the dual decomposition, in the sense of Morse theory, to the description of a handlebody as obtained by attaching 1-handles to a 0-handle.

For example, we may construct a solid torus from a torus by attaching a disc to the *meridian* and then attaching a 3-ball.

More generally, let F be a surface of genus g. Then W is obtained from F (thickened) as follows. To a maximal collection of disjoint, simple closed curves on F that do not separate F, attach discs. The boundary of the resulting manifold is a sphere. Attach a 3-ball to this. The result is a handlebody.

Thus, a handlebody may be specified by a surface together with a system of curves to which discs are to be attached.

5.2. Heegard diagrams.

Given a Heegard splitting $M = W_1 \cup W_2$, we can represent W_2 as obtained from $\partial W_2 = \partial W_1$ by attaching discs along a system of curves. Thus, a Heegard splitting for M is obtained if we are given a handlebody W_1 of genus g together with g disjoint simple curves on ∂W that do not separate ∂W. As we shall see, this is a very useful way of constructing examples.

Remark 5. Note that the system of curves specified above is *not* uniquely determined by a handlebody. The curves can be boundary curves of any *Complete Disc System* in the handlebody, i.e., a maximal family of properly embedded disjoint discs that do not separate.

Two complete disc systems are equivalent under the so-called *disc slides*, which are simply handle-slides of the discs, regarded as 2-handles.

Example 5.1 (Genus 1 Heegard splittings). By the above, a Heegard splitting of genus 1 is specified by a non-trivial simple closed curve on the boundary of a solid torus $D^2 \times S^1$. The manifold M is obtained by attaching a 2-disc to this curve and then a 3-ball.

The curves $\mu = \partial D^2 \times \{1\}$ and $\lambda = \{1\} \times S^1$ in $\partial D^2 \times S^1$ are called the meridian and the longitude. Taking the curve to which a disc is attached to be the longitude gives S^3 while taking it to be the meridian gives $S^2 \times S^1$.

The other simple curves are homotopic to $p\lambda + q\mu$, with p and q relatively prime. The corresponding manifolds are the *lens spaces* $L(p, q)$.

Note that if $F : W_1 \to W_1$ is a homeomorphism of the handlebody, then the image of a Heegard diagram under F gives another Heegard diagram for the same manifold. A particularly useful such homeomorphism is the so-called *Dehn twist*. If D is a properly embedded disc in a handlebody W, then this is the map which is equal to the identity outside a neighbourhood of D, and consists of a full twist in a neighbourhood of D.

Definition 5.1. Suppose D is a properly embedded disc in a handlebody W. Then a Dehn twist about W is a homeomorphism that is the identity outside a neighborhood $D^2 \times [-1, 1]$ of D^2 and is isotopic to $(z, t) \mapsto (ze^{\pi(t+1)}, t)$ on this neighborhood.

Example 5.2. In the case of the solid torus $D^2 \times S^1$, a Dehn twist about the properly embedded disc $D^2 \times \{1\}$ fixes the meridian and maps $\lambda \mapsto \lambda \pm 1$.

Now consider the genus 1 Heegard diagram for S^3, with a curve bounding a disc given by λ. By applying Dehn twists, we see that other Heegard diagrams of the sphere are given by attaching a disc to the curve $\lambda + k\mu, k \in \mathbb{Z}$.

Similarly, many of the Heegard diagrams constructed above correspond to the same lens space.

Exercise 8. *Show that* $L(p, q) = L(p, q + kp)$

So far, we have not even shown that the lens spaces are not all S^3. To do this, we compute the fundamental group in terms of a Heegard diagram. Recall that the fundamental group of a handlebody of genus g is a free group on g generators α_i. Each of the curves in a Heegard diagram represent a word r_i in these generators, well defined up to conjugation.

Proposition 6. *A presentation of* $\pi_1(M)$ *is given by* $\langle \alpha_1, \dots, \alpha_g; r_1, \dots, r_g \rangle$.

Proof. By the Seifert-Van Kampen theorem, attaching a 3-ball does not alter the fundamental group. Thus, it suffices to consider the manifold $\hat{M} = M \setminus B^3$.

The manifold $\hat{M}$ is obtained by attaching to a ball g 1-handles and g 2-handles. It is easy to see that $\hat{M}$ deformation retracts into a 2-complex, with a unique vertex corresponding to the ball, and edge for each 1-handle and a 2-cell for each disc. The vertex and edges form a wedge of g circles, and thus have as fundamental group the free group on g generators. Each 2-cell gives a relation. it is easy to see that the relations are as above. □

The following immediate corollary implies, for instance, that Z^4 is not a 3-manifold group (see [3] or [1]).

Corollary 7. *$\pi_1(M)$ has a presentation with an equal number of generators and relations.*

We now return to the lens spaces.

Exercise 9. $\pi_1(L(p,q)) = \mathbb{Z}/p\mathbb{Z}$

Heegard diagrams enable us to construct interesting examples, for instance homology 3-spheres. See Rolfsen [5] for more details and more examples.

Example 5.3. The Poincare homology sphere is the manifold with a Heegard splitting of genus 2 corresponding to the Heegard diagram in the figure (draw later). Using the above proposition, one can see explicitly that its fundamental group is the binary icosahedral group.

We now consider homeomorphisms of handlebodies in more detail.

Exercise 10. *Show that any orientation-preserving homeomorphism of $D^2 \times S^1$ is isotopic to the composition of Dehn twists.*

Remark 8. In the case of handlebodies of higher genus, we also need two other moves, so-called Dehn twists about annuli and handle-flips (for details, see [4]).

Thus, associated to a Heegard splitting are Heegard diagrams. Two such Heegard diagrams are equivalent under handle-slides and handlebody homeomorphisms. The more interesting question is to what extent the Heegard splitting itself is unique.

We have seen that S^3 has many Heegard splittings, but these are in some sense all derived from the simplest one. This generalises to a construction called *stabilisation.*

Definition 5.2. Let $M = W_1 \cup W_2$ be a Heegard splitting, and $\alpha \subset W_2$ a properly embedded unknotted arc. Let $W_1' = W_1 \cup \mathcal{N}(\alpha)$ and $W_2' = W_2 \setminus int(\mathcal{N}(\alpha))$. Then we say that the Heegard splitting $M = W_1' \cup W_2'$ is obtained from $M = W_1 \cup W_2$ by stabilisation.

The following fundamental theorem of Reidemeister and Singer relates two Heegard splittings of a manifold.

Theorem 9 (Reidemeister-Singer)**.** *Any two Heegard splittings of a 3-manifold M are isotopic after finitely many stabilisations.*

One would wish for more, in particular, an *a priori* bound on the number of stabilisations needed. In the case of the 3-sphere, any Heegard splitting is obtained from a unique *minimal* Heegard splitting, i.e., one that cannot be obtained by stabilising a Heegard splitting of lower genus.

It was unknown till fairly recently whether every manifold has a unique such Heegard splitting. This was shown not to be so by Casson and Gordon.

Theorem 10 (Casson-Gordon)**.** *There is a* 3*-manifold* M *which has non-isotopic, minimal Heegard splittings.*

It turns out that these Heegard splittings become isotopic after a single stabilisation. Indeed, there is no known case where more than one stabilisation is needed. This should not be regarded as an indication of what is true, but rather of our ignorance.

In the case of so called *non-Haken manifolds*, there is a bound on the number of stabilisations required, linear in the genera of the Heegard splittings, by a theorem of Rubinstein and Scharlemann.

Another interesting question is the minimum genus among Heegard splittings of a given manifold M. Since a Heegard diagram gives a presentation of $\pi_1(M)$, we see that this must be at least the rank of $\pi_1(M)$. Boileau and Zieschang have shown that there are manifolds whose Heegard genus is greater than the rank of $\pi_1(M)$.

6. More on lens spaces

We conclude by taking another look at lens spaces.

Exercise 11. *A more succinct description of the lens space* $L(p,q)$ *is as the quotient of* $S^3 \in C^2$ *by the action generated by* $(z_1,z_2) \mapsto (z_1 e^{2\pi i/p}, z_2 e^{2\pi i q/p})$*. Show that the solid tori* $H_i = \{(z_1,z_2) \in S^2 : |z_i|^2 \geq 1/2\}$ *are invariant under this action, and their images give a Heegard splitting for* $L(p,q)$*.*

Exercise 12. *Show that* $(z_1,z_2) \mapsto (z_2,z_1)$ *is a homeomorphism between* $L(p,q)$ *and* $L(p,q^{-1})$*, where* $qq^{-1} \equiv 1(mod\ p)$

Exercise 13. *As with* S^3*, lens spaces have unique minimal Heegard splittings. Thus, if* $f : L(p,q) \to L(p',q')$ *is a homeomorphism, then the image of a Heegard surface under* f *is isotopic to a Heegard surface. Use this to show that as oriented manifolds,* $L(p,q) = L(p',q')$ *iff* $p = p'$ *and* $q' \equiv q^{\pm 1}(mod\ p)$*.*

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