KNOTTED TORI

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1. Introduction

A *classical knot* is a smooth embedding of S^1 in S^3. Recall here that S^3 is $\mathbb{R}^3$ with a point added at infinity (for instance, by stereographic projection). So a knot is a thread in space with its two ends glued together.

We say two knots $K_1, K_2 \subset S^3$ are equivalent if there is an (orientation preserving) homeomorphism $f : S^3 \to S^3$ with $f(K_1) = K_2$. This is a natural extension of the idea that two spaces are the same if they are homeomorphic.

Suppose two knots are equivalent. Then their complements are homeomorphic. So the fundamental group of $S^3 - K$ is an invariant of the knot. So if we can show that the fundamental groups of two knots are different, then the knots are equivalent. We can see in this way that there are indeed plenty of knots.

In fact the fundamental group is powerful enough to tell knots apart with just a little more data - but telling whether two groups are the same is an algorithmically unsolvable problem. But this takes us far from our theme.

In these lectures we shall study smooth embeddings of the torus T^2 in S^3, i.e., *knotted tori*. As in the case of classical knots, two such embeddings T_1 and T_2 are equivalent if there is an (orientation preserving) homeomorphism $f : S^3 \to S^3$ with $f(T_1) = T_2$.

Knots give examples of knotted tori. Namely, given a knot K, we can take a tubular neighbourhood $N(K)$ of the knot. Then $T = \partial N(K)$ is an embedding of the torus in S^3.

The complement of a torus $T \in S^3$ has two components, whose closures we denote by M_1 and M_2. For equivalent tori, the (unordered) pair of spaces $\{M_1, M_2\}$ are the same. In the tori constructed from a knot, one sees that one of the complementary regions is the solid torus $D^2 \times S^1$ and the other has interior the complement of the knot K. So different knots give different knotted tori.

2. The Unknotted torus

Before studying knotted tori further, we take a closer look at the unknotted case.

Date: January 10, 2005.

2.1. **The solid torus.** The solid torus is the product $D^2 \times S^1$. It is useful to look at it from a couple of other points of view.

Firstly, one can obtain a solid torus from B^3 by *attaching a* 1*-handle*. Namely, to a pair of disjoint discs in $S^2 = \partial B^3$, glue the boundary $\{0,1\} \times D^2$ of $[0,1] \times D^2$, in such a way as to obtain an orientable manifold.

There is another way to obtain the solid torus from B^3. Take a diameter α in B^3. Delete the interior of a regular neighbourhood of this arc. It is easy to see that $B^3 \setminus int(\mathcal{N}(\alpha)) = D^2 \times S^1$.

More generally, we can take any properly-embedded *unknotted* arc $\alpha \subset B^3$, i.e., an arc α such that there exists an arc β embedded in $S^2 = \partial B^3$ and an embedded disc $E \subset B^3$ such that $\partial E = \alpha \cup \beta$. On deleting the interior of a neighbourhood of this arc, we get a solid torus.

Both these descriptions play a key role in what follows.

2.2. **The unknotted torus.** The 3-sphere S^3 is the union $S^3 = B_1 \cup B_2$ of its northern hemisphere and its southern hemisphere. Thus, S^3 is the union of two balls, with their boundaries identified using an (orientation reversing) diffeomorphism.

Let α be an unknotted properly embedded arc in B_1. Let $H_1 = B_1 \setminus int(\mathcal{N}(\alpha))$ and $H_2 = B_2 \cup \mathcal{N}(\alpha)$. Since H_1 is the result of deleting an open neighbourhood of an unknotted arc from a 3-ball, it is a solid torus. On the other hand, H_2 is obtained by adding a 1-handle to a 3-ball, and is hence also a solid torus.

Thus, S^3 is the union of two solid tori, glued along their boundary. It is easy to see that the boundary is the unknotted torus. Thus, for the unknotted torus, the closures of both the complementary components are $D^2 \times S^1$.

It is often useful, in 3-manifold topology, to think of S^3 as the unit ball in $\mathbb{C}^2$. One can see the unknotted torus from this point of view.

Exercise 1. *Let $H_i \subset S^3 \subset \mathbb{C}^2$, $i = 1, 2$, be given by $H_i = \{(z_1, z_2) \in S^2 : |z_i|^2 \geq 1/2\}$. Show that this is a decomposition of S^3 into solid tori.*

3. Knotting on the inside and outside

Consider tori embedded in $\mathbb{R}^3$. By taking the neighbouhood of a knot, it is easy to construct an embedding that is knotted. However, while such an embedding is knotted on the outside, the inside is still unknotted, i.e., the same as in the case of an unknotted torus.

With a little ingenuity, one can also construct a torus knotted on the inside but unknotted on the outside. Conceptually one can start with a torus knotted on the outside, add a point at infinity and then delete a point from the interior. to get an embedding in $\mathbb{R}^3$.

However, we shall show in these lectures that a torus cannot be knotted on both sides. To state this precisely we revert to tori embedded in S^3.

Theorem 1. *Suppose $T \subset S^3$ is a smooth embedding of a torus and M_i, $i = 1, 2$ are the closures of the complementary regions of M. Then at least one of M_1 and M_2 is homeomorphic to a solid torus.*

The proof involves some group theory, some algebraic topology and some geometric topology. Our real goal is to give a flavour of these in action. We shall then build up on these to give basic results in geometric topology.

Henceforth assume that we are given a smoothly embedded torus $T \subset S^3$ and M_i are as above. Note that $\partial M_i = T^2$ for $i = 1, 2$.

4. Group theory

Suppose $M_j = D^2 \times S^1$, then $\pi_1(M_j) = \mathbb{Z}$ and the map induced by inclusion $\iota_* : \pi_1(T) \to \pi_1(M_j)$ is not injective. Our first goal is to show that for some M_j, $\iota_{j*} : \pi_1(T) \to \pi_1(M_j)$ is not an inclusion map.

By the Van-Kampen theorem,

$$1 = \pi_1(S^3) = \pi_1(M_1) * \pi_1(M_2) / << \iota_1(h) = \iota_2(h), h \in \pi_1(T) >>$$

Suppose the maps ι_{j*} are both injections, then the fundamental group of S^3 is an *amalgamated free product*.

Definition 4.1. Suppose A, B and C are (finitely presented) groups and $\phi : C \to A$ and $\psi : C \to B$ are injective homomorphisms. Then the amalgamated free product $A *_C B$ is the group

$$A *_C B = A * B / << \phi(h) = \psi(h), h \in C >>$$

As $\pi_1(S^3) = 1$, the following general result gives a contradiction if both the maps ι_{j*} are injections.

Theorem 2. *For an amalgamated free product, the natural homomorphisms from A, B and C to $A *_C B$ are all inclusions.*

Proof. The proof of this is topological. Assume that we have connected CW-complexes X, Y and Z such that $\pi_1(X) = A$, $\pi_1(Y) = B$, and $\pi_1(Z) = C$. Assume $Z = X \cap Y$ with the maps on fundamental groups induced by inclusion being ϕ and ψ. For arbitrary groups A, B and C this can be achieved by using the mapping cylinder construction.

We shall make use of the following result whose proof is elementary algebraic topology.

Proposition 3. *Let $A \subset X$ be path connected spaces and let $\pi : \tilde{X} \to X$ be the universal covering. Then the map $\iota_* : \pi_1(A) \to \pi_1(X)$ induced by inclusion is injective if and only if each component of $\pi^{-1}(A)$ is connected.*

Let $\pi_X : \tilde{X} \to X$ be the universal cover and similarly for Y, Z. Let $W = Y \cup Z$. We construct the universal cover of W form those for Y and Z. Note that as the inclusion maps induce injections on the fundamental group, each component of $\pi_X^{-1}(Z)$ and $\pi_Y^{-1}(Z)$ is simply-connected and hence can be identified with $\tilde{Z}$ with the restrictions of π_X and π_Y being coverings.

Now take a copy of $\tilde{X}$ and to each component of $\pi_X^{-1}(Z)$ glue a copy of $\tilde{Y}$ along some component of $\pi_Y^{-1}(Z)$. For each resulting component of $\tilde{Y}$, take a copy of $\tilde{X}$ for each component of $\pi_Y^{-1}(Z)$ not already glued to $\tilde{X}$. Glue these copies to each copy of $\tilde{Y}$ along a component of $\pi_X^{-1}(Z)$. Iterate this construction taking alternately copies of $\tilde{X}$ and $\tilde{Y}$. In the limit we get a space $\tilde{W}$ which is simply connected by repeated applications of Van Kampen's theorem. The covering maps glue together to give a covering map $\pi_W : \tilde{W} \to W$, which is the universal covering of W.

By construction, the inverse images of X, Y and Z under π_W^{-1} are simply-connected, and hence the induced map into the amalgamanted free product is an injection. $\square$

5. Algebraic topology

We next see that any embedding of a torus $T \in S^3$ is *homologically unknotted*, i.e., the various homology groups of M_i's and the induced maps in homology are as in the standard case. We will only consider the case of H_1, which is what we need, but the other cases are similar.

The first step is to observe that by the Mayer-Vietoris exact sequence

$$\cdots \to H_2(S^3) \to H_1(T) \to H_1(M_1) \oplus H_1(M_2) \to H_1(S^3)$$

the map $H_1(T) \to H_1(M_1) \oplus H_1(M_2)$ is an isomorphism. As $H_1(T) = \mathbb{Z}^2$, the only possibilities are that $H_1(M_i) = \mathbb{Z}$ for $i = 1, 2$ or that $H_1(M_i) = 0$ for one of them.

In the former case the maps in homology are as in the case of the unknot. The elements of $H_1(T)$ corresponding to generators of $H_1(T)$ can be represented by a pair of simple closed curves intersecting at a single point called the meridian and the longitude. This is because any basis of $H_1(T) = \mathbb{Z}^2$ can be represented in this way.

Thus it only remains to rule out the case when (without loss of generality) $H_1(M_1) = 0$. Intuitively, the reason for this is as follows. Suppose indeed $H_1(M_1) = 0$. Then a meridinal curve μ on T bounds a compact surface F in M_1. The intersection number of the longitude λ with μ, and hence F is 1, and so λ is not null-homologous in M_1.

While this can be formalised using cup products, we shall give a different proof.

Proposition 4. *Suppose M is a compact 3-manifold with $\partial M = T$ a torus. Then $H_1(M) \neq 0$.*

Proof. Suppose $H_1(M) = 0$, then the exact sequence of the pair $(M, \partial M)$ gives $H_2(M, \partial M) = \mathbb{Z}^2$. But by Poincare duality,

$$H_2(M, \partial M) = H^1(M) = Hom(H_1(M), \mathbb{Z}) = 0$$

which gives a contradiction. □

6. Geometric topology

Our final step uses one of the most important results in three-manifold topology.

Theorem 5 (Loop theorem)**.** *Suppose M is an orientable 3-manifold and $F \subset \partial M$ is a surface. Suppose that the map induced by inclusion $\pi_1(F) \to \pi_1(M)$ is not an injection. Then there is a properly embedded disc $D \subset M$ with boundary in F such that ∂D is not homotopically trivial as a subset of ∂M.*

By the results of the previous section, at least one of M_1 and M_2, say M_1, admits such a disc D. Consider a neighbourhood N of $T \cup D$. This has two boundary components, one of which is a sphere in M_1 and the other a torus isotopic to T. The sphere bounds a ball B in M_1 (by the Schoenflies theorem in dimension 3). We see that $N \cup B$ is a solid torus whose boundary is isotopic to T and hence M_1 is homeomorphic to $N \cup B = D^2 \times S^1$.

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